Black Hole Thermodynamics and

黑洞热力学与

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Abstract

摘要

An introduction to generalized thermodynamics of quantum black holes, in the one-loop approximation, is given. The material is aimed at graduate students. The topics include quantum evaporation of black holes, Euclidean formulation

本文介绍了单圈近似下量子黑洞的广义热力学。内容面向研究生开设。主题包括黑洞的量子蒸发、欧几里得表述

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e-mail: fursaev@theor.jinr.ru of quantum theory on black hole backgrounds, the Hartle-Hawking-Israel state, generalized entropy of a quantum black hole, and its relation to the entropy of entanglement.

邮箱:fursaev@theor.jinr.ru 涵盖黑洞背景下的量子理论、哈特尔-霍金-伊斯雷尔态、量子黑洞的广义熵，以及该熵与纠缠熵的关系。

Keywords

关键词

Black holes ·Thermodynamics ·Statistical mechanics ·Quantum entanglement

黑洞·热力学·统计力学·量子纠缠

Introduction

引言

Although the perturbative quantum gravity approach has a limited range of applicability, its use in the last decades led to some conceptual issues which are to be addressed in the full-fledged quantum gravity theory. The most important issues include understanding evaporation of quantum black holes and resolution of the information loss paradox and finding a microscopic origin of black hole entropy.

尽管微扰量子引力方法的适用范围有限，但近几十年来对它的应用引出了若干概念性问题，这些问题需要在成熟的量子引力理论中得到解决。其中最重要的问题包括理解量子黑洞的蒸发、信息丢失悖论的解决，以及寻找黑洞熵的微观起源。

Black holes are specific solutions of the Einstein equations

黑洞是爱因斯坦方程的特殊解

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1}$$

which describe regions of a space-time where the gravitational field is so strong that nothing, including light signals, can escape them. The interior of a black hole is hidden from an external observer. The boundary of the unobservable region is called the horizon. In (1) we use standard notations $R_{\mu\nu}, R$, and $T_{\mu\nu}$ for the Ricci tensor, the scalar curvature, and the stress-energy tensor of matter, respectively. G is the Newton constant. The Schwarzschild and Kerr black holes are solutions to the vacuum equations (1) with $T_{\mu\nu} = 0$.

它描述了时空中引力场极强的区域: 任何事物，包括光信号，都无法从该区域逃逸。外部观测者无法观测到黑洞的内部，不可观测区域的边界被称为视界。在 (1) 式中，我们分别用标准记号 $R_{\mu\nu}, R$ 、$T_{\mu\nu}$ 表示里奇张量、标量曲率和物质的能量动量张量。G 是牛顿引力常数。史瓦西黑洞和克尔黑洞是 $T_{\mu\nu} = 0$ 条件下真空方程 (1) 的解。

In recent years our understanding of physics near the black hole horizon received important experimental evidences on the base of direct detection of gravitational waves from binary black hole mergers [1] and observations of shadows of the supermassive black holes [2].

近年来，基于对双黑洞并合产生的引力波的直接探测 [1]，以及对超大质量黑洞阴影的观测 [2]，我们对黑洞视界附近物理的理解得到了重要的实验证据支撑。

The perturbative quantum gravity, in this chapter, is treated in the one-loop approximation or as a theory of free quantum fields on black hole geometries. By explaining quantum effects near black holes in these rather restricted models, we come to important insights which have been a matter of intensive discussions in a large number of publications.

本章将在单圈近似下探讨微扰量子引力，或将其视为黑洞几何上的自由量子场理论。通过在这些受限模型中解释黑洞附近的量子效应，我们可以得到重要的启发，这些内容已经在大量文献中被广泛深入讨论。

The concrete aim of this chapter is to give a self-consistent introduction to generalized thermodynamics of quantum black holes, accessible to graduate students. The material is organized as follows. We start in section ”Necessary Definitions” with a brief description of black hole solutions by focusing mostly on the Killing structure of the black hole horizon and near-horizon features which are needed to define the first law of black hole mechanics. Quantization of free fields on external backgrounds is presented in section ”Classical Fields, Quantization, and Quasiparticles”.

本章的具体目标是为研究生提供一套自洽的量子黑洞广义热力学入门介绍。内容结构安排如下: 我们首先在“必要定义”一节简要描述黑洞解，重点聚焦黑洞视界的基灵结构，以及定义黑洞力学第一定律所需的近视界特征。“经典场、量子化与准粒子”一节介绍了外部背景下自由场的量子化。

The essence of the Hawking effect is discussed in section ”Quantum Evaporation of a Black Hole”, by using the so-called s -mode approximation. Thermodynamics of classical black holes is discussed in section ”Thermodynamic Laws of Black Holes”. The basic concept, the Hartle-Hawking-Israel state, which we use to study quantum black holes, is introduced in section ”Quantum Black Holes in Thermal Equilibrium”. We also give here some elements of a spectral theory of second-order elliptic operators and define the Euclidean effective action. From the point of view of stationary observers, quantum matter near black hole horizon is in a high-temperature regime. Hence, some features of high-temperature hydrodynamics in gravitational fields are discussed in section ”Fluid Dynamics in Gravitational Fields”. Finally, in section ”Generalized Thermodynamics of Quantum Black Holes”, we consider generalized thermodynamics of quantum black holes and, in particular, generalized black hole entropy. Quantum corrections to the entropy are discussed in detail. We introduce the notion of entanglement entropy and show that the generalized entropy is partly related to entanglement of states across the black hole horizon. Section ”Concluding Remarks” contains concluding comments.

我们在“黑洞的量子蒸发”一节利用所谓的 s 模近似讨论霍金效应的核心内容。在“黑洞热力学定律”一节讨论经典黑洞的热力学。我们用于研究量子黑洞的基本概念——哈特尔-霍金-以色列态，将在“热平衡下的量子黑洞”一节引入，我们还会在这里给出二阶椭圆算子谱理论的一些基础内容，并定义欧几里得有效作用量。从稳态观测者的视角来看，黑洞视界附近的量子物质处于高温 regime，因此我们在“引力场中的流体动力学”一节讨论引力场中高温流体动力学的若干特征。最后，我们在“量子黑洞的广义热力学”一节研究量子黑洞的广义热力学，尤其是广义黑洞熵，并详细讨论熵的量子修正。我们引入纠缠熵的概念，并说明广义熵部分与黑洞视界两侧的态纠缠有关。“结论”一节给出总结性评述。

We include in this chapter almost all required definitions and try to show how basic relations can be derived. We use the system of units where $\hbar = c = k_B = 1$ (k_B is the Boltzmann constant), the Lorentzian signature is defined as $(-, +, +, +)$, and geometrical conventions coincide with [59].

本章几乎包含了所有必要的定义，并尝试展示基本关系的推导过程。我们采用单位制 $\hbar = c = k_B = 1$ (k_B 为玻尔兹曼常数)，洛伦兹号差定义为 $(-, +, +, +)$ ，几何约定与文献 [59] 一致。

Necessary Definitions

必要定义

We start with a brief description of basic properties of black hole geometries in the near-horizon approximation. For a comprehensive introduction to black hole physics, see [32, 59]. A metric of a neutral rotating black hole, which is most interesting from the point of view of physical applications, is the Kerr solution to the Einstein equations (1) in vacuum, $T_{\mu\nu} = 0$,

我们先简要介绍近地平线近似下黑洞几何的基本性质。关于黑洞物理的全面介绍，参见 [32, 59]。从物理应用角度来看最受关注的中性旋转黑洞的度规，是真空爱因斯坦方程 (1) 的克尔解，$T_{\mu\nu} = 0$,

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{\theta\theta}d\theta^2, \tag{2}$$

$$g_{tt} = -\left(1 - \frac{2MGr}{\sum}\right),\ g_{\theta\theta} = \sum,\ g_{rr} = \frac{\sum}{\Delta}, \tag{3}$$

$$g_{t\varphi} = -\frac{2MGra}{\sum}\sin^2\theta, g_{\varphi\varphi} = \left(\left(r^2 + a^2\right)^2 - a^2\sin^2\theta\Delta\right)\frac{\sin^2\theta}{\sum}, \tag{4}$$

$$\sum = r^2 + a^2\cos^2\theta,\ \Delta = r^2 - 2MGr + a^2. \tag{5}$$

Metric (2) is written in the Boyer-Lindquist coordinates. The Kerr solution is asymptotically flat at large r . By analyzing its behavior at large r , one concludes that M is the mass of the source, and $J = Ma$ is its angular momentum. It is supposed that $MG > a$.

度规 (2) 以博耶-林德奎斯特坐标写出。克尔解在大 r 处渐近平坦。分析它在大 r 的行为后可以得出，M 是源质量，$J = Ma$ 是源角动量。我们假定 $MG > a$ 。

We also need the Schwarzschild solution, which follows from (2), (3), (4), and (5) when $a = 0$,

我们还需要施瓦西解，当 $a = 0$ 时，它可由 (2)、(3)、(4) 和 (5) 导出，

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2\left(\sin^2\theta d\varphi^2 + d\theta^2\right), \tag{6}$$

$$-g_{tt} = g_{rr}^{-1} = 1 - \frac{2MG}{r}. \tag{7}$$

We denote by $\mathcal{H}$ the event horizon of a black hole. $\mathcal{H}$ is a null hypersurface located at a constant radial coordinate r . By the definition, the normal vector l_μ to a null hypersurface is null, $l^2 = 0$. For constant r hypersurfaces $l_\mu = \delta^r_\mu$. Hence a surface $r = r_0$ is null if $g^{rr}(r_0) = 0$, or $\Delta(r_0) = 0$. This equation has two roots and $\mathcal{H}$ corresponds to the largest root, $r_0 = r_H = MG + \sqrt{(MG)^2 - a^2}$. For eternal black holes (see Fig. 1), the horizon $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ has two components, the future, $\mathcal{H}^+$, and the past $\mathcal{H}^-$ event horizons. The future light cone of any point on $\mathcal{H}^+$ is tangent to $\mathcal{H}^+$ and is directed inside the black hole. Correspondingly, past light cones on $\mathcal{H}\_$ are directed inside the white hole. A detailed discussion of this can be found in [32, 59] .

我们用 $\mathcal{H}$ 表示黑洞的事件视界。$\mathcal{H}$ 是位于恒定径向坐标 r 处的零超曲面。根据定义，零超曲面的法向量 l_μ 是类光的，即 $l^2 = 0$ 。对于恒定 r 超曲面 $l_\mu = \delta^r_\mu$ 。因此当 $g^{rr}(r_0) = 0$ 即 $\Delta(r_0) = 0$ 时，曲面 $r = r_0$ 是类光的。该方程有两个根，$\mathcal{H}$ 对应最大的那个根，即 $r_0 = r_H = MG + \sqrt{(MG)^2 - a^2}$ 。对于永恒黑洞 (参见图 1)，视界 $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ 有两个分量: 未来事件视界 $\mathcal{H}^+$ 和过去事件视界 $\mathcal{H}^-$ 。$\mathcal{H}^+$ 上任意一点的未来光锥都与 $\mathcal{H}^+$ 相切，且指向黑洞内部。相应地，$\mathcal{H}\_$ 上的过去光锥指向白洞内部。详细讨论可参见 [32, 59] 。

One can consider observers that rotate with respect to the Boyer-Lindquist coordinate grid (and therefore with respect to objects at the spatial infinity) with an angular coordinate velocity $d\varphi/dt = -g_{t\varphi}/g_{\varphi\varphi}$. An important property of (2) is that the only possible value for the angular velocity, when r approaches r_H , is

可以考虑相对于博耶-林德奎斯特坐标网 (因此也相对于空间无穷远处的物体) 以角坐标速度 $d\varphi/dt = -g_{t\varphi}/g_{\varphi\varphi}$ 转动的观测者。度规 (2) 的一个重要性质是，当 r 趋近于 r_H 时，角速度唯一可能的取值为

$$\Omega_H = \frac{a}{a^2 + r_H^2}. \tag{8}$$

The parameter Ω_H is called the angular velocity of the horizon.

参数 Ω_H 被称为视界角速度。

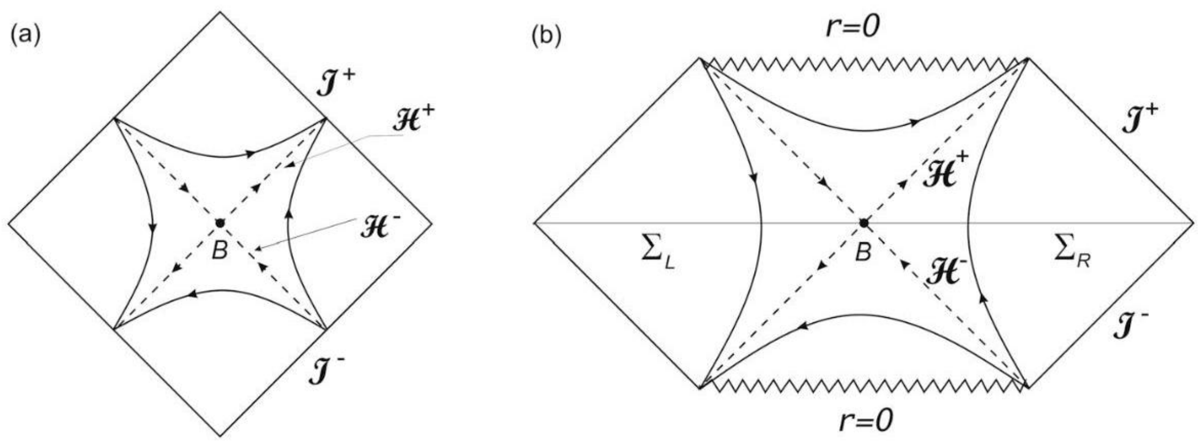

Fig. 1 Carter-Penrose diagrams for Minkowsky space-time (a) and for the eternal Schwarzschild black hole (b). Lines with arrows are integral lines of the Killing vector field. The Killing field in the Minkowsky space-time corresponds to Lorentz boosts. The Killing horizons $\mathcal{H}^{\pm}$ intersect at bifurcation 2-surfaces. A constant time section $\sum_L \cup \sum_R$ on the black hole diagram is the Einstein-Rosen bridge

图 1 闵氏时空 (a) 与永恒施瓦西黑洞 (b) 的卡特-彭罗斯图。带箭头的线是基灵矢量场的积分曲线。闵氏时空的基灵场对应洛伦兹 boost。基灵视界 $\mathcal{H}^{\pm}$ 在分叉 2 曲面相交。黑洞图上的恒定时间截面 $\sum_L \cup \sum_R$ 是爱因斯坦-罗森桥

Although Kerr solution (2), (3), (4), and (5) looks complicated, only a few features of the near-horizon geometry are required for studying quantum effects we are interested in. These features are related to the structure of time-like isometries and properties of the so-called Killing observers. A vector field $\zeta^\mu(x)$ on a manifold $\mathcal{M}$ is called a Killing field if it generates isometries of $\mathcal{M}$. The Killing field obeys the Killing equation

尽管克尔解 (2)、(3)、(4) 和 (5) 形式复杂，研究我们关注的量子效应只需要近地平线几何的少数几个性质。这些性质与类时间等距的结构，以及所谓基灵观测者的性质有关。若流形 $\mathcal{M}$ 上的矢量场 $\zeta^\mu(x)$ 生成 $\mathcal{M}$ 的等距变换，则称它为基灵场。基灵场满足基灵方程

$$\zeta_{\mu;\nu} + \zeta_{\nu;\mu} = 0. \tag{9}$$

For the Kerr solution, there is a distinguished Killing vector field, $\zeta = \partial_t - \Omega_H \partial_\varphi$, which is null on the horizon

对于克尔解，存在一个特殊的基灵矢量场 $\zeta = \partial_t - \Omega_H \partial_\varphi$ ，它在视界上是类零的

$$\zeta^2 \mid \mathcal{H} = 0 \tag{10}$$

Since $\mathcal{H}$ is the null hypersurface, Eq. (10) implies that ζ is a normal vector to $\mathcal{H}$. Properties of null hypersurfaces say that integral lines of ζ on $\mathcal{H}$ are geodesics:

由于 $\mathcal{H}$ 是零超曲面，式 (10) 表明 ζ 是 $\mathcal{H}$ 的法向量。根据零超曲面的性质，ζ 在 $\mathcal{H}$ 上的积分线是测地线:

$$\nabla_\zeta \zeta = k\zeta \tag{11}$$

One can show that $\partial_\zeta k = 0$ on $\mathcal{H}$, and, as a consequence, there is a 2D section, $\mathcal{B}$ of $\mathcal{H}$ where the Killing field is zero,

可以证明 $\partial_\zeta k = 0$ 在 $\mathcal{H}$ 上，因此存在一个二维截面，即 $\mathcal{H}$ 的 $\mathcal{B}$ ，该截面上基灵场为零，

$$\zeta^\mu \mid \mathcal{B} \in \mathcal{H} = 0. \tag{12}$$

The parameter k in (11) is called the surface gravity of the horizon, and the section $\mathcal{B}$ is called the bifurcation surface of the Killing horizons. Examples of Killing fields with bifurcating horizons are shown in Fig. 1 for the case of Minkowsky space-time and eternal Schwarzschild black hole geometry.

(11) 式中的参数 k 被称为视界的表面引力，截面 $\mathcal{B}$ 被称为基灵视界的分岔面。闵氏时空和永恒施瓦西黑洞几何中具有分岔视界的基灵场实例如图 1 所示。

The Killing field ζ is time-like in the wedge to the right from $\mathcal{H}^+$ and $\mathcal{H}^-$. In this region one can define a frame of reference of observers whose 4-velocities u^μ are directed along ζ ,

基灵场 ζ 在 $\mathcal{H}^+$ 和 $\mathcal{H}^-$ 右侧的楔形区域内是类时的。在该区域中可以定义观测者的参考系，其四维速度 u^μ 沿 ζ 方向，

$$u^\mu = \zeta^\mu / \sqrt{B},\ B = -\zeta^2 = -g_{tt} - 2\Omega_H g_{t\varphi} - \Omega_H^2 g_{\varphi\varphi}. \tag{13}$$

Such observers are called the Killing observers. For a rotating black hole, the given frame of reference is defined in a domain close to $\mathcal{H}$, where $B > 0$. The congruence of the trajectories is specified [46] by the acceleration w^μ , the rotation tensor $A_{\mu\nu}$, and the local angular velocity $\Omega(x)$

这类观测者被称为基灵观测者。对于旋转黑洞，上述参考系定义在靠近 $\mathcal{H}$ 的区域，该区域满足 $B > 0$ 。轨迹汇集由加速度 w^μ 、旋转张量 $A_{\mu\nu}$ 和局部角速度 $\Omega(x)$ 给出 [46],

$$w^\mu = \nabla_u u^\mu,\ A_{\mu\nu} = \frac{1}{2} h^\lambda_\mu h^\rho_\nu \left(\nabla_\rho u_\lambda - \nabla_\lambda u_\rho\right),\ \Omega(x) = \left(\frac{1}{2} A_{\mu\nu} A^{\mu\nu}\right)^{1/2}, \tag{14}$$

where $h^\lambda_\mu = \delta^\lambda_\mu + u_\mu u^\lambda$. Quantities (14) appear under study of quantum systems in thermal equilibrium with the black hole, see section ”High-Temperature

其中 $h^\lambda_\mu = \delta^\lambda_\mu + u_\mu u^\lambda$ 。量 (14) 出现在与黑洞处于热平衡的量子系统研究中，参见“高温

Asymptotic”. Local angular velocity determines rotation of the Killing frame with respect to a local inertial frame.

渐近”章节。局部角速度决定了基灵系相对于局部惯性系的转动。

One can use (11) to relate the definition of the surface gravity to the strength of gravity near the horizon

可以利用 (11) 将表面引力的定义与视界附近的引力强度联系起来

$$k = \lim_{r \to r_H} \left(\sqrt{B w^2}\right) = \frac{r_H - MG}{a^2 + r_H^2}. \tag{15}$$

The right-hand side (r.h.s.) of (15) follows from (3), (4), (5), and (13).

(15) 的右侧可由 (3)、(4)、(5) 和 (13) 得到。

It is convenient to change in the Boyer-Lindquist coordinates φ to $\varphi' = \varphi - \Omega_H t$. In the new coordinates the Killing vector field is $\zeta = \partial_t$, that is, the Killing observers do not move with respect to the new coordinate grid. Metric (2) can be rewritten as

我们可以方便地将博耶-林德奎斯特坐标 φ 更换为 $\varphi' = \varphi - \Omega_H t$ 。在新坐标下，基灵矢量场为 $\zeta = \partial_t$ ，也就是说基灵观测者相对于新坐标网格不发生运动。度规 (2) 可以改写为

$$ds^2 = -B\left(dt + a_i dx^i\right)^2 + h_{ij} dx^i dx^j, \tag{16}$$

where $x^i = (r, \varphi', \theta), a_i dx^i = a_\varphi d\varphi'$. The nonvanishing components of acceleration and rotation are $w_i = (\ln B)_{,i}/2, A_{ij} = \sqrt{B}\left(a_{i,j} - a_{j,i}\right)/2$.

其中 $x^i = (r, \varphi', \theta), a_i dx^i = a_\varphi d\varphi'$ 。加速度和转动的非零分量为 $w_i = (\ln B)_{,i}/2, A_{ij} = \sqrt{B}\left(a_{i,j} - a_{j,i}\right)/2$ 。

One can check that at small $r - r_H$

可以验证，当 $r - r_H$ 很小时

$$B \simeq 4k^2 h r_H (r - r_H), h_{rr} \simeq \frac{h r_H}{r - r_H} + O\left((r - r_H)^2\right),$$

$$h(\theta) \equiv \frac{\sum(r_H, \theta)}{2r_H\sqrt{(MG)^2 - a^2}}. \tag{17}$$

Here we took into account that $h_{rr} = g_{rr}$. It is convenient to introduce a new coordinate ρ :

此处我们考虑了 $h_{rr} = g_{rr}$ 的性质。引入新坐标 ρ 会更方便:

$$r - r_H \simeq \frac{\rho^2}{4r_H} \tag{18}$$

connected with the proper distance L to the horizon

它与到视界的固有距离 L 相关

$$L(r, \theta) = \int_{r_H}^{r} dr' \sqrt{h_{rr}} \simeq \sqrt{h}\rho. \tag{19}$$

In the leading approximation $B \simeq hk^2\rho^2$. Since the local angular velocity vanishes near the horizon, $\Omega = O(\rho)$, terms $a_i dx^i$ in (16) can be neglected near $\mathcal{H}$.

领头阶近似下有 $B \simeq hk^2\rho^2$ 。由于局部角速度在视界附近趋于零，即 $\Omega = O(\rho)$ ，因此 (16) 中的 $a_i dx^i$ 项在 $\mathcal{H}$ 附近可以忽略。

One comes to the following form of near-horizon black hole metric (2):

可得到如下形式的近视界黑洞度规 (2):

$$ds^2 \simeq h(\theta)\left(-k^2\rho^2 dt^2 + d\rho^2\right) + dl^2, \tag{20}$$

$$dl^2 = \gamma_{\theta\theta} d\theta^2 + \gamma_{\varphi\varphi}(d\varphi')^2, \tag{21}$$

where $\gamma_{\alpha\beta} = h_{\alpha\beta}(r = r_H)$ and (21) is the metric on $\mathcal{B}$. It can be shown that $\gamma_{\theta\theta} = g_{\theta\theta}(r_H), \gamma_{\varphi\varphi} = g_{\varphi\varphi}(r_H)$. Therefore the space-time near $\mathcal{H}$ has the product structure $R^2 \times \mathcal{B}$. In case of the Schwarzchild black hole $\mathcal{B}$ is $S^2, \mathcal{B}$ is closed and has the topology of S^2 for the Kerr solution, and $\mathcal{B} = R^2$ for flat space-time.

其中 $\gamma_{\alpha\beta} = h_{\alpha\beta}(r = r_H)$ ，式 (21) 为 $\mathcal{B}$ 上的度规。可以证明 $\gamma_{\theta\theta} = g_{\theta\theta}(r_H), \gamma_{\varphi\varphi} = g_{\varphi\varphi}(r_H)$ 。因此 $\mathcal{H}$ 附近的时空具有乘积结构 $R^2 \times \mathcal{B}$ 。对于史瓦西黑洞，$\mathcal{B}$ 是 $S^2, \mathcal{B}$ ；克尔解中该流形是闭合的，拓扑结构为 S^2 ；平直时空下其拓扑结构为 $\mathcal{B} = R^2$ 。

The area $\mathcal{A}$ of $\mathcal{B}$,

$\mathcal{B}$ 的面积 $\mathcal{A}$,

$$\mathcal{A}=\int d\theta d\varphi\sqrt{\det\gamma_{\alpha\beta}}=4\pi\left(r_H^2+a^2\right), \tag{22}$$

is called the area of black hole horizon. It plays an important role in the subsequent discussions. To get (22), one should use (3) and (4).

被称为黑洞视界面积。它在后续讨论中发挥重要作用。要得到式 (22)，需用到式 (3) 和式 (4)。

Classical Fields, Quantization, and Quasiparticles

经典场、量子化与准粒子

Since stationary black holes have a universal structure near the horizon, it is worth studying classical fields in this region by using, as an example, a free scalar field ϕ . The field equation is

由于稳态黑洞在视界附近具有普适结构，我们以自由标量场 ϕ 为例研究该区域的经典场。场方程为

$$\left(\nabla^2-m^2\right)\phi=\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)-m^2\phi=0, \tag{23}$$

where m is the mass of the field. The analysis of wave equations like (23) on the Kerr background can be found in [21]. According to (20), when r close r_H , Eq. (23) reduces to

其中 m 是该场的质量。克尔背景下类似 (23) 的波动方程分析可参见文献 [21]。根据式 (20)，当 r 接近 r_H 时，式 (23) 简化为

$$\left[-\frac{1}{k^2\rho^2}\partial_t^2+\partial_\rho^2\right]\phi-L\phi=0. \tag{24}$$

Here $L=h\left(\Delta+m^2\right)$ and Δ is the Laplace operator on $\mathcal{B}$. Solutions to (24), which are interpreted by a Killing observer as excitations with energies ω , are

此处 $L=h\left(\Delta+m^2\right)$ 和 Δ 是 $\mathcal{B}$ 上的拉普拉斯算子。式 (24) 的解被基林观察者解释为能量为 ω 的激发，形式为

$$i\partial_t\phi=\omega\phi \tag{25}$$

If $\psi_\lambda(\theta,\varphi)$ is an eigen-function of L with an eigenvalue λ , the solution to (24) and (25) can be written as

若 $\psi_\lambda(\theta,\varphi)$ 是 L 对应本征值 λ 的本征函数，则式 (24) 和 (25) 的解可写为

$$\phi(t,\rho,\theta,\varphi)=e^{-i\omega t}\phi_\omega(x)\,\psi_\lambda(\theta,\varphi), \tag{26}$$

$$\left(-\partial_x^2+V_\lambda(x)\right)\phi_\omega(x)=\omega^2\phi_\omega(x). \tag{27}$$

Here $\rho = Ce^{kx}$ and $V_\lambda(x) = \lambda C^2 k^2 e^{2kx}$, where C is a constant. Note that $\mathcal{B}$ is compact and h is finite on $\mathcal{B}$. One can show that eigenvalues λ of L are discrete and positive.

此处 $\rho = Ce^{kx}$ 和 $V_\lambda(x) = \lambda C^2 k^2 e^{2kx}$ ，其中 C 为常数。注意 $\mathcal{B}$ 是紧致的，且 h 在 $\mathcal{B}$ 上有限。可以证明，L 的本征值 λ 是离散且正的。

The spectrum of ω , called the single-particle energies, is defined by a Schroedinger-like equation (27) with an effective potential V_λ . We assume that $C > 0$ and imply that limit $\rho \to 0$ corresponds to $x \to -\infty$. A straightforward conclusion from (27) is that fields near the horizon are effectively massless and the spectrum of ω is continuous. Effects related to nonzero mass m or properties of L are exponentially suppressed.

ω 的谱被称为单粒子能量，由带有有效势 V_λ 的类薛定谔方程 (27) 定义。我们假设 $C > 0$ ，由此可知极限 $\rho \to 0$ 对应 $x \to -\infty$ 。由式 (27) 可直接得到结论: 视界附近的场实际是无质量的，且 ω 的谱是连续谱。非零质量 m 或 L 性质相关的效应是指数抑制的。

As an illustration, we give an exact form of V_λ for the Schwarzschild black hole ($a = 0$)

作为示例，我们给出史瓦西黑洞 ($a = 0$) 对应的 V_λ 的精确形式

$$V_\lambda(x) = B(r)\left(m^2 + \frac{\partial_r B(r)}{r} + \frac{l(l+1)}{r^2}\right), x = \int^r \frac{dr'}{B(r')}, \tag{28}$$

$B(r) = 1 - r_H/r$. Eigenvalues $\lambda = l(l+1)$ belong to the spectrum of a Laplacian on unit 2-sphere. At $r \to r_H$ (28) reduces to (27). Near the horizon, $x \to -\infty$, the potential is exponentially small. The mass of the field dominates, $V_\lambda(x) \simeq m^2$, as $x \to +\infty$. Potential $V_\lambda(x)$ reaches a maximum at some point x whose position depends on l .

$B(r) = 1 - r_H/r$ 。本征值 $\lambda = l(l+1)$ 属于单位二维球面上拉普拉斯算子的谱。在 $r \to r_H$ 处，式 (28) 退化为式 (27)。在视界附近 $x \to -\infty$ ，势是指数小量。当 $x \to +\infty$ 时，场质量占主导，即 $V_\lambda(x) \simeq m^2$ 。势 $V_\lambda(x)$ 在某点 x 处取得最大值，该点的位置取决于 l 。

Since we are interested in near-horizon physics, the above analysis suggests a serious technical simplification: One can focus only on dynamics in the (t, r) coordinate sector where all fields are effectively massless.

由于我们研究的是视界附近的物理，上述分析提示了一个重要的技术简化: 我们只需关注 (t, r) 坐标区的动力学，该区域内所有场实际都是无质量的。

For further purposes we recall basic elements of quantum field theory on classical backgrounds; for details see [41]. Take, as an example, the free scalar field with equation (23). For a pair of solutions, f_1, f_2 , to (23), one introduces a relativistic inner product

为便于后续讨论，我们回顾经典背景下量子场论的基本要素；细节参见文献 [41]。以满足方程 (23) 的自由标量场为例。对于满足式 (23) 的一组解 f_1, f_2 ，引入相对论内积

$$\langle f_1, f_2 \rangle = \int_\Sigma d\sum\nolimits^\mu i\left(f_1^\star \partial_\mu f_2 - \partial_\mu f_1^\star f_2\right), \tag{29}$$

which is taken on a Cauchy hypersurface $\sum$. The choice of $\sum$ is unimportant since (29) conserves under variations of $\sum$. Suppose that $f_i^{(\pm)}$ are solutions enumerated by a certain set of indices i , discrete or continuous, and normalized as

其定义在柯西超曲面 $\sum$ 上。由于式 (29) 在 $\sum$ 变化下守恒，因此 $\sum$ 的选择不影响结果。假设 $f_i^{(\pm)}$ 是由某组离散或连续的指标 i 编号的解，且按如下方式归一化:

$$\left\langle f_i^{(\pm)}, f_j^{(\pm)} \right\rangle = \pm\delta_{ij},\ \left\langle f_i^{(+)}, f_j^{(-)} \right\rangle = 0. \tag{30}$$

Suppose also that any solution to (23) can be written as a linear combination

再假设式 (23) 的任意解都可以写为线性组合

$$\phi(x) = \sum_i a_i f_i^{(+)}(x) + \sum_j b_j^\star f_j^{(-)}(x). \tag{31}$$

Quantization procedure implies that ϕ, a_i , and $b_j^\star$ are replaced with operators $\hat{\phi}, \hat{a}_i$, and $\hat{b}_j^+$ with the following commutation relations:

量子化过程要求将 ϕ, a_i 和 $b_j^\star$ 替换为算符 $\hat{\phi}, \hat{a}_i$ 和 $\hat{b}_j^+$ ，并满足如下对易关系:

$$\left[\hat{a}_i, \hat{a}_j^+\right] = \delta_{ij}, \left[\hat{b}_i, \hat{b}_j^+\right] = \delta_{ij}. \tag{32}$$

Operators $\hat{a}_j^+, \hat{b}_j^+$ create quasiparticles which, in general, may not carry any definite energy. In stationary space-times with the Killing field $\zeta = \partial_t$ there is a special set of single-particle modes such as (compare with (25))

算符 $\hat{a}_j^+, \hat{b}_j^+$ 产生准粒子，一般来说，这些准粒子不一定带有确定能量。在具有 Killing 场 $\zeta = \partial_t$ 的稳态时空中，存在一组特殊的单粒子模式，例如 (与式 (25) 比较)

$$i\partial_t f_i^{(\pm)} = \pm\omega_i^{(\pm)} f_i^{(\pm)}. \tag{33}$$

ω_i are positive numbers which we call single-particle energies. Correspondingly, the Killing observers interpret $f_i^{(\pm)}$ as excitations with certain energies. The spectrum of single-particle energies can be found from equations like Eq. (27).

ω_i 是我们称之为单粒子能量的正数。相应地，Killing 观测者将 $f_i^{(\pm)}$ 阐释为具有确定能量的激发。单粒子能量的能谱可以通过式 (27) 这类方程求得。

In General Relativity the definition of a particle depends on the frame of reference where observations are done. To see how different choices of creation and annihilation operators are connected, suppose that ϕ is a real field. One has two decompositions

在广义相对论中，粒子的定义依赖于观测所在的参考系。为了说明不同的产生和湮灭算符选择之间如何关联，假设 ϕ 是一个实场。我们有两种分解

$$\hat{\phi}(x) = \sum_i \left(\hat{a}_i f_i^{(+)}(x) + \hat{a}_i^+ f_i^{(-)}(x)\right) = \sum_j \left(\hat{\bar{a}}_j \bar{f}_j^{(+)}(x) + \hat{\bar{a}}_j^+ \bar{f}_j^{(-)}(x)\right) \tag{34}$$

corresponding to different definitions of quasiparticles, $\hat{a}_i, \hat{\bar{a}}_j$. The reality condition implies that $f_i^{(-)} = \left(f_i^{(+)}\right)^\star, \bar{f}_i^{(-)} = \left(\bar{f}_i^{(+)}\right)^\star$. By using normalization conditions (30), one finds from (34) that

对应 $\hat{a}_i, \hat{\bar{a}}_j$ 不同的准粒子定义。实性条件给出 $f_i^{(-)} = \left(f_i^{(+)}\right)^\star, \bar{f}_i^{(-)} = \left(\bar{f}_i^{(+)}\right)^\star$ 。利用归一化条件 (30), 可以从式 (34) 得到

$$\hat{a}_i = \left\langle f_i^{(+)}, \hat{\phi} \right\rangle = \sum_j \left(c_{ij}\hat{\bar{a}}_j + d_{ij}\hat{\bar{a}}_j^+\right), \tag{35}$$

$$c_{ij} = \left\langle f_i^{(+)}, \bar{f}_j^{(+)} \right\rangle, \; d_{ij} = \left\langle f_i^{(+)}, \bar{f}_j^{(-)} \right\rangle. \tag{36}$$

Coefficients c_{ij} and d_{ij} are called the Bogoliubov coefficients. As follows from (35), the average number of quasiparticles created by $\hat{a}_i$ is nontrivial, in general,

系数 c_{ij} 和 d_{ij} 被称为博戈留波夫系数。由式 (35) 可知，一般情况下，由 $\hat{a}_i$ 产生的准粒子平均数是非平凡的,

$$N_i = \langle \bar{0} | \hat{a}_i^+ \hat{a}_i | \bar{0} \rangle = \sum_j |d_{ij}|^2, \tag{37}$$

in the vacuum state $|\bar{0}\rangle$ which does not contain particles of the other sort, $\hat{\bar{a}}_j|\bar{0}\rangle = 0$.

在真空态 $|\bar{0}\rangle$ which does not contain particles of the other sort, $\hat{\bar{a}}_j|\bar{0}\rangle = 0$ 中。

Quantum Evaporation of a Black Hole

黑洞的量子蒸发

We present now a sketch of arguments demonstrating the Hawking effect of quantum evaporation of a black hole [44]. The effect is related to physics near the black hole horizon where, according to section "Necessary Definitions", the geometry has the universal form, see Eq. (20). In the leading approximation the dynamics occurs in (t, r) or (t, ρ) coordinate sector, while dependence on angles θ and φ is unimportant.

我们现在概述证明黑洞量子蒸发霍金效应的论证过程 [44]。该效应与黑洞视界附近的物理性质相关，根据“必要定义”一节，该处几何具有通用形式，参见式 (20)。在领头近似下，动力学过程发生在 (t, r) 或 (t, ρ) 坐标区，而对角度 θ 和 φ 的依赖无关紧要。

For simplicity we consider the behavior of quantum scalar field ϕ outside a spherically symmetric star which collapses and creates a Schwarzschild black hole. We assume that ϕ does not depend on the angles, that is, ϕ is the so-called s -mode.

为简化推导，我们考虑球对称坍缩形成史瓦西黑洞的恒星外部，量子标量场 ϕ 的行为。我们假设 ϕ 不依赖角度，即 ϕ 是所谓的 s 模。

Since mass of the field is not important near the horizon, we also assume that ϕ is massless.

由于在场的质量在视界附近并不重要，我们进一步假设 ϕ 是无质量的。

The wave equation (23) reduces to $\nabla^2\phi = 0$, where ∇^2 is the operator on a 2D space-time $\mathcal{M}_2$. Outside the star,

波动方程 (23) 可约化为 $\nabla^2\phi = 0$ ，其中 ∇^2 是二维时空 $\mathcal{M}_2$ 上的算符。在恒星外部，

$$ds^2 = -B(r)\,dt^2 + \frac{dr^2}{B(r)},\ B(r) = 1 - \frac{r_H}{r}. \tag{38}$$

A standard approach is to introduce ingoing (v, r) and outgoing (u, r) Eddington-Finkelstein coordinates

标准方法是引入内向 (v, r) 和外向 (u, r) 爱丁顿-芬克斯坦坐标

$$u = t - x, v = t + x, x = \int^r \frac{dr'}{B(r')}, \tag{39}$$

$$ds^2 = -Bdv^2 + 2dvdr = -Bdu^2 - 2dudr = -B(r)\,dudv. \tag{40}$$

Null coordinates u and v are retarded and advanced times, respectively. Lines of constant u or v are radial rays. Past-directed null geodesics end up on the past null infinity denoted by $\mathcal{I}^-$, and future-directed null geodesics end up on the future null infinity $\mathcal{I}^+$. These infinities can be used as parts of the Cauchy surface $\sum$ where the relativistic product (29) is defined. In computations $\mathcal{I}^+$ and $\mathcal{I}^-$ are replaced, respectively, with null surfaces $v = C > 0$ or $u = -C < 0$, with large C . For an eternal black hole shown in Fig. 1, coordinates (v, r) are continued across $\mathcal{H}^+$. inside the black hole, and the outgoing coordinates can be continued across $\mathcal{H}^-$.

零坐标 u 和 v 分别是推迟时间和超前时间，恒定 u 或 v 的线是径向射线。指向过去的零类测地线终止于标记为 $\mathcal{I}^-$ 的过去零无穷远，指向未来的零类测地线终止于未来零无穷远 $\mathcal{I}^+$ 。这两个无穷远可作为定义相对论内积 (29) 的柯西面 $\sum$ 的组成部分。计算中，$\mathcal{I}^+$ 和 $\mathcal{I}^-$ 分别被大 C 处的零曲面 $v = C > 0$ 或 $u = -C < 0$ 替代。对于图 1 所示的永恒黑洞，坐标 (v, r) 可以延拓穿过 $\mathcal{H}^+$ 进入黑洞内部，外向坐标也可以延拓穿过 $\mathcal{H}^-$ 。

2D massless scalar fields are easy to analyze since $\mathcal{M}_2$ is conformally flat. The metric can be brought to the form $ds^2 = -e^{2\sigma}dudv$, and a general solution to the wave equation

二维无质量标量场易于分析，因为 $\mathcal{M}_2$ 是共形平坦的。度规可以化为 $ds^2 = -e^{2\sigma}dudv$ 的形式，波动方程的通解

$$\nabla^2\phi = -4e^{-2\sigma}\partial_u\partial_v\phi = 0 \tag{41}$$

is a combination of left-moving and right-moving modes, $\phi(u,v)=\phi_R(u)+\phi_L(v)$.

是左行模和右行模的组合，即 $\phi(u,v)=\phi_R(u)+\phi_L(v)$。

Before we proceed with solutions on black hole geometries, it is instructive to consider fields on 2D Minkowski space-time with metric (40), where $B=1, x=r$, and $r>0$. A complete set of modes is

在继续讨论黑洞几何上的解之前，我们不妨先考虑度规为 (40) 的二维闵氏时空中的场，其中 $B=1, x=r$ ，且 $r>0$。模的完备集为

$$f_\omega^{(+)}(u,v)=\frac{1}{\sqrt{4\pi\omega}}\left(e^{-i\omega v}-e^{-i\omega u}\right),\left\langle f_\omega^{(+)},f_\sigma^{(+)}\right\rangle=\delta(\omega-\sigma). \tag{42}$$

Since r is a radial coordinate with the center at $r=0$, each mode is a combination of left-moving and right-moving waves with the Dirichelt boundary condition, $f_\omega^{(+)}=0$, at $r=0$. In these coordinates waves coming from $\mathcal{I}^-$ are reflected from the center and travel, unchanged, to $\mathcal{I}^+$, see Fig. 2. Due to this reflection condition, null infinities $\mathcal{I}^\pm$ are the Cauchy surfaces where the relativistic inner product $\left\langle f_\omega^{(+)},f_\sigma^{(+)}\right\rangle$ can be defined as

由于 r 是以 $r=0$ 为中心的径向坐标，每个模都是满足 $r=0$ 处狄利克雷边界条件 $f_\omega^{(+)}=0$ 的左行波和右行波的组合。在这些坐标中，来自 $\mathcal{I}^-$ 的波从中心反射后不变地传播到 $\mathcal{I}^+$ ，参见图 2。由于该反射条件，零无穷远 $\mathcal{I}^\pm$ 就是可定义相对论内积 $\left\langle f_\omega^{(+)},f_\sigma^{(+)}\right\rangle$ 的柯西面，其内积为

$$\langle f_1,f_2\rangle=i\int_{\mathcal{I}^-}dv\left(f_1^\star\partial_v f_2-\partial_v f_1^\star f_2\right)=i\int_{\mathcal{I}^+}du\left(f_1^\star\partial_u f_2-\partial_u f_1^\star f_2\right). \tag{43}$$

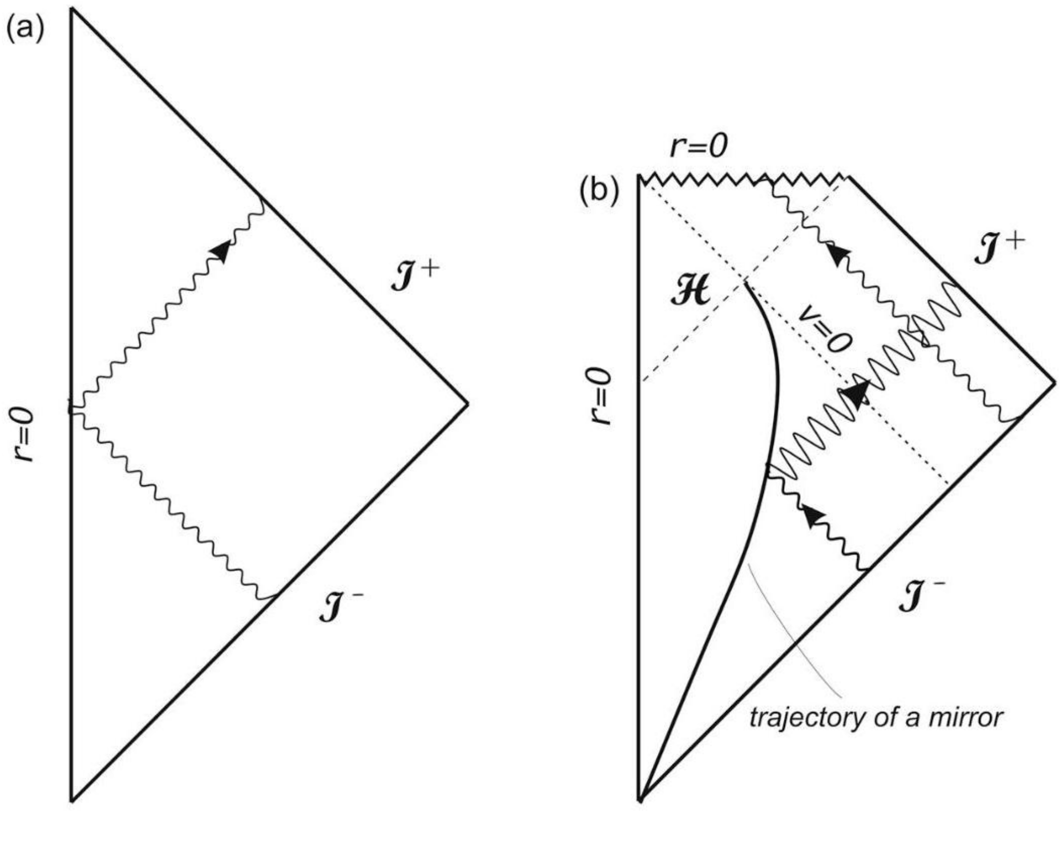

Fig. 2 Carter-Penrose diagrams that show propagation of waves from $\mathcal{I}^-$ in Minkowsky spacetime (a) and in space-time of a collapsing star (bb). In flat space-time all modes from $\mathcal{I}^-$ reach $\mathcal{I}^+$ unchanged. The right figure shows the "escape" modes $\phi_{\rm esc}(u,v)$ defined at $v<0$. These modes leave the star just before the horizon is formed and behave as if reflected from an imaginary mirror. Modes $\phi_H(v)$ are shown as waves from $\mathcal{I}^-$ at $v>0$. They cross $\mathcal{H}$ and end on the singularity $r=0$

图 2 卡特-彭罗斯图，展示了来自 $\mathcal{I}^-$ 的波在闵可夫斯基时空 (a) 与坍缩恒星时空 (bb) 中的传播。在平直时空中，来自 $\mathcal{I}^-$ 的所有模式都保持不变地到达 $\mathcal{I}^+$ 。右图展示了定义在 $v<0$ 处的“出射”模式 $\phi_{\rm esc}(u,v)$ 。这些模式在视界形成前刚刚离开恒星，表现得就像是从虚镜子反射而来。模式 $\phi_H(v)$ 被表示为 $v>0$ 处来自 $\mathcal{I}^-$ 的波。它们穿过 $\mathcal{H}$ ，最终终止于奇点 $r=0$

Formula (43) also holds on space-times with metric $ds^2=-e^{2\sigma}dudv$; it does not depend on σ .

公式 (43) 同样适用于度规为 $ds^2=-e^{2\sigma}dudv$ 的时空；它不依赖于 σ 。

There is a principle distinction between fields near a collapsing star and fields in the flat space-time: For the star there are ingoing waves, $\phi_H(v)$, which cannot escape to $\mathcal{I}^+$. Such waves come after the black hole horizon is formed, cross $\mathcal{H}$, and move inside the black hole. Schematically the 2D part of a collapsing star is shown in Fig. 2. A general solution to (41) in the null coordinates is a combination of left- and right-moving modes of the following form:

坍缩恒星附近的场与平直时空中的场存在本质区别: 对于恒星，存在入射波 $\phi_H(v)$ ，它们无法逃逸到 $\mathcal{I}^+$ 。这类波在黑洞视界形成后产生，穿过 $\mathcal{H}$ ，进入黑洞内部。图 2 概略展示了坍缩恒星的二维部分。零坐标下方程 (41) 的通解是以下形式的左行模式与右行模式的组合:

$$\phi(u,v)=\phi_{\rm esc}(u,v)+\phi_H(v)\,. \tag{44}$$

By $\phi_{\rm esc}(u,v)$ we denote waves that can escape to $\mathcal{I}^+$. Like waves in the Minkowsky space-time, $\phi_{\rm esc}(u,v)$ satisfy a sort of reflection condition. If $v=0$ is the trajectory of the last ingoing ray which escapes the black hole, just before the horizon is formed, $\phi_{\rm esc}(u,v),\phi_H(v)$ are defined for $v<0$ or $v>0$, respectively. For solutions (44) the equivalent Cauchy surfaces are $\mathcal{I}^-$, in far past, or $\mathcal{I}^+\cup\mathcal{H}$, in future.

我们用 $\phi_{\rm esc}(u,v)$ 表示能够逃逸到 $\mathcal{I}^+$ 的波。与闵可夫斯基时空中的波类似，$\phi_{\rm esc}(u,v)$ 满足一类反射条件。设 $v=0$ 是视界形成前最后一道入射射线能逃逸出黑洞的轨迹，则 $\phi_{\rm esc}(u,v),\phi_H(v)$ 分别对 $v<0$ 和 $v>0$ 有定义。对于解 (44)，等价柯西面分别是遥远过去的 $\mathcal{I}^-$ 和未来的 $\mathcal{I}^+\cup\mathcal{H}$ 。

To determine $\phi_{\rm esc}$, one can trace modes inside the collapsing star and find the "boundary condition." This option is complicated and requires additional model assumptions. Another strategy is to define $\phi_{\rm esc}$ by requiring its regularity across $\mathcal{H}$. The fact that $\phi_{\rm esc}$ cannot be taken as waves (42) in flat space-time is clear since the outgoing coordinate u is not analytical on $\mathcal{H}$.

为了确定 $\phi_{\rm esc}$ ，可以在坍缩恒星内部追踪模式并得到“边界条件”，但这种方法复杂度高，还需要额外的模型假设。另一种方法是通过要求 $\phi_{\rm esc}$ 在 $\mathcal{H}$ 上正则来定义它。$\phi_{\rm esc}$ 不能作为平直时空下的波 (42)，这一点很明确，因为出射坐标 u 在 $\mathcal{H}$ 上不是解析的。

There are alternative coordinates, the Kruskal coordinates,

存在另一种替代坐标，即克鲁斯卡尔坐标,

$$U = U(u) = -ae^{-ku},\ V = V(v) = be^{kv}, \tag{45}$$

which both are analytic on $\mathcal{H}$. Here a, b are some positive constants which can be chosen such that $ds^2 \simeq -dUdV$ near $\mathcal{H}$. The event horizon $\mathcal{H}$ is defined by conditions $U = 0, v > 0$. It is important that U and V can be interpreted near $\mathcal{H}$ as retarded and advanced times for freely falling observers.

它在 $\mathcal{H}$ 上处处解析。此处 a, b 是若干正的常数,可以通过选择它们使得在 $\mathcal{H}$ 附近满足 $ds^2 \simeq -dUdV$ 。事件视界 $\mathcal{H}$ 由条件 $U = 0, v > 0$ 定义。重要的是，对于自由下落观测者，U 和 V 在 $\mathcal{H}$ 附近可以分别被解释为推迟时间和超前时间。

In the black hole exterior, $U > 0$, and U can be continued inside the black hole, where $U < 0$. Therefore, for example, the following modes:

在黑洞外部，$U > 0$ 和 U 可以延拓到黑洞内部，而在那里 $U < 0$ 。因此，举个例子，以下模式:

$$f^{(+)}_{\text{in},\omega}(u, v) = \frac{1}{\sqrt{4\pi\omega}}\left(e^{-i\omega v} - e^{-i\omega U(u)}\right),\ -\infty < u, v < \infty \tag{46}$$

behave well at $\mathcal{H}$. These modes:

在 $\mathcal{H}$ 处表现良好。这些模式:

(i) Are ingoing waves $e^{-i\omega v}$ with frequency ω as measured by freely moving observers in the far past, near $\mathcal{I}^-$

(i) 是入射波 $e^{-i\omega v}$ ，具有远过去靠近 $\mathcal{I}^-$ 处自由运动观测者测量得到的频率 ω

(ii) Are outgoing waves $e^{-i\omega U}$ with frequency ω for freely falling observers near $\mathcal{H}$

(ii) 对于接近 $\mathcal{H}$ 的自由下落观测者而言，是频率为 ω 的出射波 $e^{-i\omega U}$

(iii) Make a complete set on the Cauchy surfaces $\mathcal{I}^-$ or $\mathcal{I}^+ \cup \mathcal{H}$

(iii) 在柯西曲面 $\mathcal{I}^-$ 或 $\mathcal{I}^+ \cup \mathcal{H}$ 上构成完备集

Therefore for the collapsing star, a quantum state that is experienced as a vacuum by freely falling observers, including observers near $\mathcal{I}^-$, should be modeled by a state $|\Psi\rangle$ determined by the condition:

因此对于坍缩恒星，一个被包括接近 $\mathcal{I}^-$ 处观测者在内的自由下落观测者感知为真空的量子态，应当由满足如下条件的态 $|\Psi\rangle$ 建模:

$$\hat{a}_\omega|\Psi\rangle = 0, \tag{47}$$

where operators $\hat{a}_\omega$ are introduced by (31) with respect to modes $f^{(+)}_{\mathrm{in},\omega}$ defined in (46).

其中算符 $\hat{a}_\omega$ 由式 (31) 针对式 (46) 定义的模 $f^{(+)}_{\mathrm{in},\omega}$ 引入

The ϕ_H part of modes (46) is simply proportional to $e^{-i\omega v}$ for $v > 0$. The escape part, $\phi_{\mathrm{esc}}, v < 0$, looks as a solution to the wave equation in the presence of a perfectly reflecting accelerated mirror which moves along the trajectory

对于 $v > 0$，式 (46) 中模的 ϕ_H 部分简单正比于 $e^{-i\omega v}$ 。出射部分 $\phi_{\mathrm{esc}}, v < 0$ 可看作沿如下轨迹运动的理想反射加速镜存在时波动方程的解

$$v = U(u), \tag{48}$$

see Fig. 2. This fictitious mirror plays the role of a strong gravity, and it transforms ingoing waves $e^{-i\omega v}$ in the escape part to outgoing waves $e^{-i\omega U(u)}$.

参见图 2。这个假想镜子扮演强引力的角色，它将出射部分的入射波 $e^{-i\omega v}$ 转换为出射波 $e^{-i\omega U(u)}$

It is well known that accelerated mirrors create particles. Analogously the collapsing body creates the flux of the Hawking radiation. To see this define modes

众所周知加速镜会产生粒子。类似地，坍缩天体也会产生霍金辐射通量。据此定义如下模:

$$f^{(+)}_{\mathrm{out},\sigma}(u, v) = \frac{1}{\sqrt{4\pi\sigma}} \left(e^{-i\sigma u} - e^{-i\sigma \bar{v}(v)}\right), \; -\infty < u < \infty, v < 0, \tag{49}$$

where $\bar{v}(v) = -\frac{1}{k} \ln\left(-\frac{v}{a}\right)$. These out-modes:

其中 $\bar{v}(v) = -\frac{1}{k} \ln\left(-\frac{v}{a}\right)$。这些出模:

(i) Are reduced to outgoing waves $e^{-i\omega u}$ with frequency ω as measured by freely moving observer in the far future, near $\mathcal{J}^+$

(i) 对于远未来接近 $\mathcal{J}^+$ 处的自由运动观测者测量而言，退化为频率为 ω 的出射波 $e^{-i\omega u}$

(ii) Satisfy the Dirichlet condition on ”trajectory” (48)

(ii) 满足“轨迹”(48) 上的狄利克雷边界条件

(iii) Do not have $\phi_H(v)$ part

(iii) 不存在 $\phi_H(v)$ 部分

(iv) Make a complete set with the same normalization on the null surfaces discussed above

(iv) 在上述讨论的类光面上具有相同归一化，构成完备集

Define creation and annihilation operators $\hat{b}^{+}$ and $\hat{b}$ for out-modes $f^{(+)}_{\text{out},\sigma}$. The corresponding quasiparticles are interpreted by observers near $\mathcal{I}^{+}$ as particles with certain energies. One identifies these particles with the Hawking quanta.

为出模 $f^{(+)}_{\text{out},\sigma}$ 定义产生和湮灭算符 $\hat{b}^{+}$ 与 $\hat{b}$ 。接近 $\mathcal{I}^{+}$ 处的观测者将相应准粒子解释为具有确定能量的粒子，这些粒子就是霍金量子。

The number of the Hawking quanta in a given state is, see Eqs. (36) and (37),

给定态中霍金量子的数量参见式 (36) 与 (37)，为

$$N_{\sigma}=\langle\Psi\left|\hat{b}^{+}_{\sigma}\hat{b}_{\sigma}\right|\Psi\rangle=\int d\omega|d_{\sigma,\omega}|^{2}, \tag{50}$$

$$d_{\sigma,\omega}=\left\langle f^{(+)}_{\text{out},\sigma},f^{(-)}_{\text{in},\omega}\right\rangle=-\frac{1}{2\pi}\sqrt{\frac{\sigma}{\omega}}\int_{-\infty}^{\infty}du e^{i\sigma u+i\omega U(u)}. \tag{51}$$

The product in (51) is defined on $\mathcal{I}^{+}$. Integral in r.h.s. of (51) can be performed after the substitution $t=e^{-ku}$

(51) 中的乘积定义在 $\mathcal{I}^{+}$ 上。代入 $t=e^{-ku}$ 后即可计算 (51) 右侧的积分

$$d_{\sigma,\omega}=-\frac{1}{2\pi k}\sqrt{\frac{\sigma}{\omega}}\exp\left(-\frac{\pi\sigma}{2k}\right)\Gamma\left(-\frac{i\sigma}{k}\right)(\omega a)^{\frac{i\sigma}{k}}, \tag{52}$$

where $\Gamma(z)$ is the Γ -function

其中 $\Gamma(z)$ 是 Γ 函数

$$\Gamma(z)=\int_{0}^{\infty}dt t^{z-1}e^{-t} \tag{53}$$

By using the property $|\Gamma(iz)|^{2}=\pi/(z\sinh\pi z)$, one gets the number of the Hawking quanta in the form:

利用性质 $|\Gamma(iz)|^{2}=\pi/(z\sinh\pi z)$ ，可以得到霍金量子的数量形式如下:

$$N_{\sigma}=\frac{\Gamma_{\sigma}}{e^{\sigma/T_{H}}-1}. \tag{54}$$

Coefficients Γ_{σ} are called the gray-body factors. In the considered case Γ_{σ} is a constant which includes a regularized integral over ω .

系数 Γ_{σ} 被称为灰体因子。在本文讨论的情形中，Γ_{σ} 是一个包含了对 ω 的正则化积分的常数。

A remarkable fact is that the Hawking quanta are distributed according to Planck's law with a temperature

一个值得注意的结论是，霍金量子服从普朗克分布，对应的温度

$$T_H = \frac{k}{2\pi} \tag{55}$$

called the Hawing temperature.

称为霍金温度。

The simplified analysis presented here can be extended beyond the s -mode approximation and near-horizon approximation. Massless modes that depend on angles will experience a partial reflection on the potential $V_l(r)$, see (28). This effect yields nontrivial factors Γ_σ .

本文给出的简化分析可以推广到 s 模近似和近视界近似之外。依赖角度的无质量模会在势场 $V_l(r)$ 处发生部分反射，参见式 (28)。该效应会产生非平凡因子 Γ_σ 。

The Hawking effect can be interpreted as a process of creation of particle-antiparticle pairs in the strong gravitational field near the horizon. Antiparticles created in this process tunnel inside the black hole, while particles make the Hawking flux. Semiclassical estimations of the tunneling probability [61] are in agreement with (54), see [77] for a review.

霍金效应可以被解释为视界附近强引力场中产生正反粒子对的过程: 该过程产生的反粒子隧穿进入黑洞内部，粒子则形成霍金辐射流。隧穿概率的半经典估计 [61] 与式 (54) 一致，综述可见文献 [77]。

If, after the collapse of the star, the black hole evaporates completely, it results in violation of the unitarity and information loss since the initial pure state $|\psi\rangle$ evolves to a mixed thermal state. This paradox, despite several interesting hypotheses [76], has not been resolved so far.

如果恒星坍缩后黑洞完全蒸发，由于初始纯态 $|\psi\rangle$ 演化为混合热态，这会导致幺正性破缺与信息丢失。这一悖论尽管已有多个有意思的假说 [76]，至今仍未得到解决。

Thermodynamic Laws of Black Holes

黑洞热力学定律

Black Hole Mechanics

黑洞力学

If a black hole appears as a result of the gravitational collapse of a star, it quickly reaches a stationary state characterized by a certain mass M and an angular momentum J . By using purely classical Einstein equations or on the base of definitions of (8), (15), and (22), one arrives at the following variational formula [4]:

如果黑洞由恒星引力坍缩形成，它会迅速达到由质量 M 和角动量 J 描述的定态。仅利用经典爱因斯坦方程，或基于式 (8)、(15) 和 (22) 的定义，可以推导出如下变分公式 [4]:

$$\delta M = T_H \delta S^{BH} + \Omega_H \delta J, \tag{56}$$

where T_H is given by (55) and

其中 T_H 由式 (55) 给出，且

$$S^{BH} = \frac{1}{4G}\mathcal{A}. \tag{57}$$

$\mathcal{A}$ is the surface area of the horizon (see above), and G is the Newton gravitational constant.

$\mathcal{A}$ 是视界的表面积 (见上文)，G 是牛顿引力常数。

The quantity S^{BH} was introduced in $[8-10,44]$ and is called the Bekenstein-Hawking entropy.

物理量 S^{BH} 由 $[8-10,44]$ 提出，被称为贝肯斯坦-霍金熵。

Relation (56) has the form of the first law of thermodynamic where S^{BH} has the meaning of an entropy, T_H is a temperature, and M is an internal energy. If the collapsing matter was not electrically neutral, a black hole has an additional parameter, an electric charge Q . Then the r.h.s. of (56) would acquire an additional term $\Phi_H \delta Q$, where Φ_H is the difference of the electric potential at the horizon and at infinity. M, J, Q are the only parameters a black hole in the Einstein-Maxwell theory can have. Its metric in the most general case is the Kerr-Newmann metric. This statement is known as the ”no-hair” theorem, see, e.g., [32].

关系式 (56) 符合热力学第一定律的形式，其中 S^{BH} 为熵，T_H 为温度，M 为内能。如果坍缩物质不是电中性的，黑洞会多出一个参数——电荷 Q 。此时式 (56) 的右侧会增加一项 $\Phi_H \delta Q$ ，其中 Φ_H 是视界与无穷远之间的电势差。M, J, Q 是爱因斯坦-麦克斯韦理论中黑洞仅有的参数。最一般情况下其度规为克尔-纽曼度规。这个结论就是著名的“无毛”定理，参见例如文献 [32]。

The Bekenstein-Hawking entropy is one of the most mysterious quantities in black hole thermodynamics. For supermassive black holes with masses of the order of 10^9 solar masses, S^{BH} is of the order of 10^{95} ; it is eight orders of magnitude larger than the entropy of the microwave background radiation in the visible part of the universe. This raises a natural question about microscopic degrees of freedom whose number is consistent with the Bekenstein-Hawking entropy.

贝肯斯坦-霍金熵是黑洞热力学中最神秘的物理量之一。对于质量约为 10^9 倍太阳质量的超大质量黑洞，S^{BH} 约为 10^{95} ；比可见宇宙中微波背景辐射的熵高出八个数量级。这自然引出一个问题: 什么样的微观自由度的数量能和贝肯斯坦-霍金熵匹配。

The reason why this question is fundamental is because it goes beyond the black hole physics itself. On the one hand, its answer may give important insights into the as yet mysterious nature of quantum gravity. On the other hand, since the thermodynamics of black holes is a low-energy phenomenon, understanding of the black hole entropy may be possible without knowing details of quantum gravity, for example, in the framework of the perturbative quantum gravity methods.

这个问题具有根本性，因为它早已超出黑洞物理学本身的范畴。一方面，它的答案可能为我们至今仍未能理解的量子引力本质提供重要启发；另一方面，由于黑洞热力学是低能现象，即使不了解量子引力的细节也有可能理解黑洞熵，例如可以在微扰量子引力方法的框架下开展研究。

Black Holes and Euclidean Theory

黑洞与欧氏理论

There is another way to see that black holes look like thermodynamic systems. Consider the partition function of a quantum system with a (normally ordered) Hamiltonian $:\hat{H}:$ At temperature $T = \beta^{-1}$,

我们可以通过另一种方法看出黑洞具有热力学系统的性质。考虑一个拥有 (正规序) 哈密顿量 $\hat{H}$ 的量子系统的配分函数: 在温度 $T = \beta^{-1}$ 下,

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta :\hat{H}:} \tag{58}$$

As is known,(58) can be interpreted as a trace of the evolution operator $\hat{U}(t) = e^{-it\hat{H}}$ with imaginary time interval $t = -i\beta$. This allows one (see discussion in next sections) to represent $Z(\beta)$ as a path integral in the corresponding Euclidean quantum theory with periodic or antiperiodic boundary conditions in the Euclidean time $\tau = it$. In application to black holes in vacuum, this implies that instead of the Lorentzian Ricci flat solutions, $R_{\mu\nu} = 0$, see (1), we should consider analogous solutions on Riemannian or Euclidean manifolds with the signature +, +, +, +. Such solutions are called gravitational instantons.

众所周知，(58) 可以被解释为演化算符 $\hat{U}(t) = e^{-it\hat{H}}$ 在虚时间区间 $t = -i\beta$ 下的迹。由此我们可以将 $Z(\beta)$ 表示为对应欧氏量子理论中的路径积分，该路径积分在欧氏时间 $\tau = it$ 上满足周期或反周期边界条件 (相关讨论见后续章节)。将该方法应用于真空中的黑洞时，这意味着我们不需要考虑洛伦兹型里奇平坦解 $R_{\mu\nu} = 0$ (见式 (1))，而应当考虑符号为 (+, +, +, +) 的黎曼流形或欧氏流形上的类似解。这类解被称为引力瞬子。

Consider the Kerr solution (2), (3), and (5). The corresponding instanton can be obtained from (2),(3), and (5) by the Wick rotation of time, $t = -i\tau$, and by changing a to $-ia$ to ensure that non-diagonal term $2g_{t\varphi}dtd\varphi$ in metric (2) remains real. (It should be noted that quantum theory which we discuss in next sections does not require the Wick rotation of a , so one can work in principle with complex metrics.) By using parameters of the Lorentzian solution, r_H, Ω_H, k , one defines analogous parameters $\bar{r}_H = r_H(ia)$, $\overline{\Omega}_H = \Omega_H(ia)$, $\bar{k} = k(ia)$ for the instanton. The Euclidean Kerr solution has analogous symmetries. The Killing vector field $\zeta = \partial_\tau + \overline{\Omega}_H \partial_\varphi$ does not have a horizon, as in the Lorentzian theory, but it has fixed points located on a closed 2D surface $r = \bar{r}_H$, the Euclidean horizon, which we denote by $\mathcal{B}$, in the same way as the bifurcation surface. Near $\mathcal{B}$ the Euclidean metric looks as follows (compare with (20)):

考虑克尔解 (2)、(3) 和 (5)，我们可以对时间做威克转动 $t = -i\tau$ ，并将 a 替换为 $-ia$ 以保证度规 (2) 中的非对角项 $2g_{t\varphi}dtd\varphi$ 保持为实，从而得到对应的瞬子。(需要注意的是，我们后续章节讨论的量子理论并不要求对 a 做威克转动，因此原则上可以直接在复度规下计算。) 利用洛伦兹解的参数 r_H, Ω_H, k ，我们可以为瞬子定义对应的参数 $\bar{r}_H = r_H(ia), \overline{\Omega}_H = \Omega_H(ia), \bar{k} = k(ia)$ 。欧氏克尔解具有类似的对称性。与洛伦兹理论不同，基灵矢量场 $\zeta = \partial_\tau + \overline{\Omega}_H\partial_\varphi$ 不存在视界，但它存在位于闭合二维曲面 $r = \bar{r}_H$ 上的不动点，该曲面就是欧氏视界，我们仿照分岔面的记法将其记为 $\mathcal{B}$ 。在 $\mathcal{B}$ 附近，欧氏度规形式如下 (与式 (20) 对比):

$$ds^2 \simeq \bar{h}(\theta)\left(\bar{k}^2\rho^2 d\tau^2 + d\rho^2\right) + dl^2,\ 0 < \tau \le \beta. \tag{59}$$

At arbitrary periodicity (59) has a conical singularity at $\rho = 0\,(r = \bar{r}_H)$. The singularity disappears if

在任意周期性下，(59) 在 $\rho = 0\,(r = \bar{r}_H)$ 处存在锥奇点。当满足以下条件时，该奇点消失:

$$\beta = \beta_H = \frac{2\pi}{\bar{k}}. \tag{60}$$

Thus, the regularity condition requires that the period β_H coincides with the inverse Hawking temperature (55).

因此，正则性条件要求周期 β_H 与霍金温度的倒数 (55) 一致。

The fact that the Euclidean horizon is a fixed point set of the Killing field associated with time translations also implies a ”thermodynamic” form of the gravity action on the black hole instanton. It is easy to check that the Euclidean action, $I_E[\phi]$, say, for a scalar field ϕ , between a constant time hypersurface $\sum_\tau$ and a hypersurface $\sum_{\tau'}$ with $\tau' = \tau + \beta$ has the form $I_E[\phi] = \beta H$, where H has a form of the Hamiltonian of the system. On static solutions, $\partial_\tau\phi = 0, H$ coincides with the energy of the system.

欧氏视界是时间平移对应的基灵场的不动点集，这一性质也说明黑洞瞬子上的引力作用具有“热力学”形式。不难验证，对于标量场 ϕ ，在固定时间超曲面 $\sum_\tau$ 和满足条件 $\tau' = \tau + \beta$ 的超曲面 $\sum_{\tau'}$ 之间，欧氏作用量 $I_E[\phi]$ 的形式为 $I_E[\phi] = \beta H$ ，其中 H 具有系统哈密顿量的形式。在静态解上，$\partial_\tau\phi = 0, H$ 与系统的能量一致。

Consider now the Einstein-Hilbert action on Riemannian (Euclidean) manifolds $\mathcal{M}$

现在考虑黎曼 (欧氏) 流形 $\mathcal{M}$ 上的爱因斯坦-希尔伯特作用量

$$I_E[g] = -\frac{1}{16\pi G}\left[\int_{\mathcal{M}} d^4x\sqrt{g}R + 2\int_{\partial\mathcal{M}} d^3y\sqrt{h}K\right]. \tag{61}$$

The last term in the r.h.s. of (61) should be added, according to Gibbons and Hawking [42], when $\mathcal{M}$ has a boundary $\partial\mathcal{M}$. This term depends on the trace K of the extrinsic curvature of $\partial\mathcal{M}$, and it guarantees that variations $I_E[g]$ do not contain variations of normal derivatives of the metric on $\partial\mathcal{M}$. To avoid infrared divergences in (61) on asymptotically flat space-times, $I_E[g]$ is defined with a subtraction of the corresponding action on a flat space-time.

根据吉本斯和霍金的结论 [42]，当 $\mathcal{M}$ 存在边界 $\partial\mathcal{M}$ 时，应当添加 (61) 式右侧的最后一项。该项依赖于 $\partial\mathcal{M}$ 外曲率的迹 K，它保证变分 $I_E[g]$ 不包含 $\partial\mathcal{M}$ 上度量法向导数的变分。为避免渐近平直时空中 (61) 式出现红外发散，$I_E[g]$ 通过减去平时空的对应作用量来定义。

If the Killing field ∂_τ does not have fixed points, the only boundaries of constant τ hypersurfaces belong to $\partial\mathcal{M}$. Then action (61) computed between $\sum_\tau$ and $\sum_{\tau+\beta}$ has the structure, $I_E = \beta H$, where H is a Hamiltonian. On asymptotically flat solutions, after the subtraction in $I_E[g]$, H coincides with the ADM mass [47].

若 Killing 场 ∂_τ 没有不动点，则常 τ 超曲面的所有边界都属于 $\partial\mathcal{M}$。此时在 $\sum_\tau$ 与 $\sum_{\tau+\beta}$ 之间计算得到的作用量 (61) 具有结构 $I_E = \beta H$，其中 H 是哈密顿量。对于渐近平直解，完成减法后，$I_E[g]$, H 中的结果与 ADM 质量一致 [47]。

Situation is different for black hole instantons since ∂_τ has fixed points on $\mathcal{B}$. As a result, $\sum_\tau$ ends on $\mathcal{B}$, and the Euclidean horizon becomes an internal boundary. To calculate the action on $\mathcal{M}$, one should consider the near-horizon part $\mathcal{M}_\varepsilon$ of $\mathcal{M}$ separately. The metric of $\mathcal{M}_\varepsilon$ can be approximated by (59) with the outer boundary located at $\rho = \varepsilon$. The gravitational action taken on a black hole instanton then becomes [42, 47]

黑洞瞬子的情况不同，因为 ∂_τ 在 $\mathcal{B}$ 上存在不动点。结果就是 $\sum_\tau$ 终止于 $\mathcal{B}$，欧几里得视界成为内边界。要计算 $\mathcal{M}$ 上的作用量，需要单独考虑 $\mathcal{M}$ 的近视界部分 $\mathcal{M}_\varepsilon$。$\mathcal{M}_\varepsilon$ 的度量可以用 (59) 式近似，外边界位于 $\rho = \varepsilon$。黑洞瞬子的引力作用量于是可写为 [42, 47]

$$I_E = \beta_H (M - \Omega_H J) - \frac{\mathcal{A}}{4G}, \tag{62}$$

where $\mathcal{A}$ is the area of $\mathcal{B}$. The first term in the r.h.s. of (62) appears from a domain outside of $\mathcal{M}_\varepsilon$ where constant time hypersurfaces do not have intersections. The last term, $-\mathcal{A}/(4G)$, appears from the boundary term

其中 $\mathcal{A}$ 是 $\mathcal{B}$ 的面积。(62) 式右侧的第一项来自 $\mathcal{M}_\varepsilon$ 外侧的区域，该区域内常时间超曲面没有交点。最后一项 $-\mathcal{A}/(4G)$ 来自边界项

$$\lim_{\varepsilon\to 0} \frac{1}{8\pi G} \int_{\partial\mathcal{M}_\varepsilon} d^3y\sqrt{h}K = \frac{\mathcal{A}}{4G}, \tag{63}$$

see discussion of this point, based on topological arguments, in [3].

基于拓扑论证对该问题的讨论参见文献 [3]。

It follows from (62) that I_E/β_H can be interpreted as the free energy of a thermodynamic system with energy M, entropy $\mathcal{A}/(4G)$, and temperature $1/\beta_H$, in accord with the first law (56).

由 (62) 式可得，I_E/β_H 可以解释为能量为 M、熵为 $\mathcal{A}/(4G)$、温度为 $1/\beta_H$ 的热力学系统的自由能，这与第一定律 (56) 一致。

It will be important for the future discussion to point out that the thermodynamic form of the action (62) can be extended to the case when the Euclidean time τ in the instanton solution has an arbitrary period β. Such a geometry is not regular because of conical singularities at $\mathcal{B}$ with the deficit angle $2\pi(1 - \beta/\beta_H)$. For this reason, it is not a solution of the Einstein equations near $\mathcal{B}$. The black hole thermodynamics at $\beta \neq \beta_H$ is called an off-shell approach. Components of the Riemann tensor on manifolds with conical singularities behave as distributions at $\mathcal{B}$, see [39]. In particular, the integral curvature is

需要指出的是，对后续讨论而言很重要的一点: 作用量 (62) 的热力学形式可以推广到瞬子解中欧几里得时间 τ 具有任意周期 β 的情况。这种几何由于在 $\mathcal{B}$ 处存在亏缺角为 $2\pi(1-\beta/\beta_H)$ 的锥奇点，并不正则，因此它不是 $\mathcal{B}$ 附近爱因斯坦方程的解。在 $\beta \neq \beta_H$ 下研究黑洞热力学的方法被称为脱壳法。带锥奇点流形上的黎曼张量分量在 $\mathcal{B}$ 处表现为分布，参见文献 [39]。具体来说，积分曲率为

$$\int_{\mathcal{M}} d^4x\sqrt{g}R = 4\pi\left(1-\frac{\beta}{\beta_H}\right)\mathcal{A} + \int_{\mathcal{M}/\mathcal{B}} d^4x\sqrt{g}R, \tag{64}$$

where $\mathcal{M}/\mathcal{B}$ is the regular part of $\mathcal{M}$. By using (63) and (64), it is not difficult to check that the off-shell action

其中 $\mathcal{M}/\mathcal{B}$ 是 $\mathcal{M}$ 的正则部分。利用 (63) 和 (64) 不难验证，脱壳作用量

$$I_E = \beta(M-\Omega_H J) - \frac{\mathcal{A}}{4G} = I_E(\beta, M, J) \tag{65}$$

holds the thermodynamic form. The advantage of the off-shell formulation is that β is a free parameter which is not related to the mass M and angular momentum $J = Ma$ of a black hole. If $I_E/\beta = F$ is interpreted as a free energy, the Bekenstein-Hawking entropy can be derived by using statistical-mechanical formula:

满足热力学形式。脱壳表述的优势在于 β 是一个自由参数，与黑洞的质量 M 和角动量 $J = Ma$ 无关。若将 $I_E/\beta = F$ 诠释为自由能，即可利用统计力学公式推导出贝肯斯坦-霍金熵:

$$S^{BH} = \beta^2\partial_\beta F(\beta, M, J)_{\beta=\beta_H} = \left(\beta\partial_\beta - 1\right) I_{E,\beta=\beta_H}. \tag{66}$$

We use the definition (66) in what follows.

下文我们均使用定义 (66)。

To avoid unnecessary complications, boundary conditions for a black hole have not been specified in the above analysis. The importance of boundary conditions has been pointed out in [83] for the case of a Schwarzschild black hole inside a spherical cavity of a finite radius. It can be shown by using the Gibbons-Hawking action that such a black hole at a certain size of the cavity behaves as a stable thermodynamic system.

为避免不必要的复杂性，上述分析并未指定黑洞的边界条件。文献 [83] 已经指出，对于有限半径球形空腔内部的史瓦西黑洞，边界条件十分重要。利用吉本斯-霍金作用量可以证明，当空腔尺寸取特定值时，这类黑洞可表现为稳定的热力学系统。

Asymptotically anti-de Sitter black hole solutions in the Einstein gravity with a negative cosmological constant have a similar property [49]: They can be in a stable equilibrium state when their size is greater than the radius of the anti-de Sitter space.

带负宇宙学常数的爱因斯坦引力中的渐近反德西特黑洞解具有类似性质 [49]: 当这类黑洞的尺寸大于反德西特空间的半径时，它们可以处于稳定平衡态。

Thermodynamics of black holes in higher dimensional gravity theories with the negative cosmological constant is used [80] to study finite-temperature gauge theories in the framework of the so-called AdS/CFT correspondence [62, 81]. Interestingly, charged black holes in anti-de Sitter space-times have a phase structure similar to that of the van der Waals-Maxwell liquid-gas systems in a space-time of one-dimensional lower [20].

在所谓的 AdS/CFT 对应框架下 [62, 81]，带负宇宙学常数的高维引力理论中的黑洞热力学被用于研究有限温度规范理论 [80]。有趣的是，反德西特时空中的带电黑洞的相结构，与低一维时空中范德华-麦克斯韦气液系统的相结构相似 [20]。

Black Holes in Generalized Gravity Theories

推广引力理论中的黑洞

It should be noted that variational formulas analogous to (56) can be found for asymptotically flat black hole solutions in a general classical theory of gravity arising from a diffeomorphism invariant Lagrangian [79]. In these theories one comes to a generalization of the Bekenstein-Hawking formula (57). Moreover, the black hole entropy can be interpreted as a Noether charge associated with the horizon Killing field ζ . As an example, consider a modification of the Einstein gravity by terms quadratic in curvatures. The bulk part of gravity action (on Euclidean manifolds) is

需要注意的是，对于由微分同胚不变拉格朗日量导出的一般经典引力理论中的渐近平坦黑洞解，同样可以得到类似 (56) 的变分公式 [79]。在这些理论中，我们得到了贝肯斯坦-霍金公式 (57) 的推广形式。此外，黑洞熵可以解释为与视界基灵场 ζ 相关的诺特荷。例如，考虑爱因斯坦引力被曲率二次项修正后的理论，(欧几里得流形上) 引力作用量的体部分为

$$I\left[g,(G,\Lambda,c_i)\right]=-\int d^4x\sqrt{g}\left[\frac{R}{16\pi G}-2\Lambda-c_1R^2-c_2R_{\mu\nu}R^{\mu\nu}-c_3R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\right],$$

(67)

where $R_{\alpha\beta\mu\nu}$ and $R_{\mu\nu}$, and R are the Riemann and Ricci tensors and the scalar curvature, respectively. Introduce two normal vectors n_i^μ at the bifurcation surface $\mathcal{B},\left(n_i\cdot n_j\right)=\delta_{ij}$, and define at $\mathcal{B}$ the following invariants:

其中 $R_{\alpha\beta\mu\nu}$ 、$R_{\mu\nu}$ 和 R 分别是黎曼张量、里奇张量和标量曲率。在分岔面 $\mathcal{B},\left(n_i\cdot n_j\right)=\delta_{ij}$ 引入两个法向量 n_i^μ ，并在 $\mathcal{B}$ 处定义如下不变量:

$$R_{ii}=R_{\mu\nu}n_i^\mu n_j^\nu,\ R_{ijij}=R_{\mu\nu\lambda\rho}n_i^\mu n_j^\nu n_i^\lambda n_j^\rho. \tag{68}$$

The entropy of stationary black holes in such theories is

这类理论中稳态黑洞的熵为

$$S^{BH}(G, c_i) = \frac{1}{4G}\mathcal{A} - \int_{\mathcal{B}} d^2x\sqrt{\gamma}\left(8\pi c_1 R + 4\pi c_2 R_{ii} + 8\pi c_3 R_{ijij}\right), \tag{69}$$

where $\gamma = \det \gamma_{\alpha\beta}$, and $\gamma_{\alpha\beta}$ is defined in (21).

其中 $\gamma = \det \gamma_{\alpha\beta}$ 和 $\gamma_{\alpha\beta}$ 由式 (21) 定义。

An alternative way to derive (69) is to use the off-shell approach or the conical singularity method. As was shown in [39], integrals of powers of the curvature tensor can be well defined on off-shell instantons with conical singularities in the linear approximation in the deficit angle. Such integrals are similar to (64) and yield (69) if one applies (66) for definition of the entropy.

推导式 (69) 的另一种方法是使用离壳方法或锥奇点法。正如文献 [39] 所示，在亏缺角的线性近似下，曲率张量的幂次积分在带锥奇点的离壳瞬子上是良定义的。这类积分与式 (64) 形式相似，若用式 (66) 定义熵就可以得到式 (69)。

Generalized Second Law

广义第二定律

By considering classical processes with black holes, one can conclude that the area of the horizon never decreases, the observation which is reminiscent to the second law. Black hole must have an intrinsic entropy proportional to the horizon area; otherwise, processes like a gravitational collapse would be at odds with the second law. The second law of thermodynamic in the presence of black holes can be written in the generalized form

通过研究黑洞的经典过程，可以得出结论: 黑洞视界的面积永远不会减小，这一结论让人联想到热力学第二定律。黑洞必然存在与视界面积成正比的内禀熵；否则，引力坍缩这类过程就会和第二定律矛盾。存在黑洞时，热力学第二定律可以写成如下广义形式

$$\delta S^{BH} + \delta S_m \geq 0, \tag{70}$$

which states that the sum of the Bekenstein-Hawking entropy of a black hole and the entropy of a surrounding matter does not decrease in physical processes.

该定律指出，在物理过程中，黑洞的贝肯斯坦-霍金熵与周围物质熵之和永不减小。

The generalized second law (70) poses a number of serious questions when it is applied to quantum matter around black holes. The very definition of S_m , when the black hole horizon serves as a boundary of the system, requires clarification. Thermal entropy of a relativistic plasma is not well defined near $\mathcal{H}$ due to an infinite blue-shift of the energies of quanta in this region. The classical part, S^{BH} , of the generalized entropy depends on the gravitational coupling G which participates in the renormalization of the ultraviolet divergences. Can S_m in (70) be considered as a quantum correction to the classical entropy S^{BH} ?

当将广义第二定律 (70) 应用于黑洞周围的量子物质时，产生了许多严肃的问题。当黑洞视界作为系统的边界时，S_m 本身的定义仍需要澄清。相对论性等离子体的热熵在 $\mathcal{H}$ 附近无法良好定义，因为该区域量子能量会发生无限蓝移。广义熵的经典部分 S^{BH} 依赖于引力耦合常数 G，后者参与紫外发散的重整化。式 (70) 中的 S_m 可以被视为经典熵 S^{BH} 的量子修正吗?

Although such questions may look technical, their resolution is a necessary step toward understanding profound conceptual issues brought in theoretical physics by black holes. Perturbative quantum gravity is a testbed where these questions can be dealt with by using conventional quantum field theory.

尽管这些问题看起来是技术性的, 但解决它们是理解黑洞给理论物理学带来的深刻概念问题的必要一步。微扰量子引力提供了一个试验场，我们可以在这里利用传统量子场论处理这些问题。

Quantum Black Holes in Thermal Equilibrium

热平衡中的量子黑洞

The Hartle-Hawking-Israel State

哈特尔-霍金-以色列态

To proceed with the discussion of generalized second law (70) in case of quantum fields around a black hole, one needs to specify the quantum state of the system. If a black hole has an astrophysical mass, it evaporates due to the Hawking effect very slowly, as if being in thermal equilibrium with the radiation. A real equilibrium state can be realized for an eternal black hole placed inside a cavity with perfectly reflecting walls. A classical analog of this system has been studied in [83].

为了讨论黑洞周围量子场情况下的广义第二定律 (70)，需要明确系统的量子态。若黑洞具有天体物理质量，它会因霍金效应极缓慢地蒸发，就如同与辐射处于热平衡中。真正的平衡态可以通过将永恒黑洞置于拥有完美反射壁的空腔中实现。该系统的经典类比已在文献 [83] 中研究。

Consider, for simplicity, quantum fields on space-time of an eternal Schwarzschild black hole. This black hole has two space-like singularities, the future (black hole) singularity and the past (white) hole singularities, and two asymptotically flat, left and right regions, separated by the horizons, see Fig. 1. The horizons $\mathcal{H}^{\pm}$ bifurcate at a two-sphere $\mathcal{B}$. The structure of the Killing field of the Schwarzschild black hole is almost identical to that of Minkowsky space-time. This fact indicates that, for a black hole, there may be defined a quantum state which is a counterpart of the Minkowsky vacuum. Such a state does exist and is called the Hartle-Hawking-Israel state (HHI state) [43,52]. Stationary observers that move along integral lines of ∂_t are analogous to the Rindler observers, and they see the HHI state as a thermal bath at the Hawking temperature.

为简化讨论，我们考虑永恒史瓦西黑洞时空上的量子场。该黑洞存在两个类空奇点: 未来 (黑洞) 奇点和过去 (白洞) 奇点，还有被视界分隔的左右两个渐近平直区域，参见图 1。视界 $\mathcal{H}^{\pm}$ 在二维球面 $\mathcal{B}$ 处分叉。史瓦西黑洞的基林场结构与闵氏时空几乎完全相同。这一事实表明, 对于黑洞, 可以定义一个对应于闵氏真空的量子态。这样的态确实存在，被称为哈特尔-霍金-以色列态 (HHI 态)[43,52]。沿 ∂_t 积分线运动的稳态观测者等效于伦德勒观测者，他们会观测到 HHI 态是霍金温度下的热浴。

To come to the definition of this equilibrium state, we use the results of section ”Black Holes and Euclidean Theory”. The fact that classical gravity formulated on gravitational instantons has a thermodynamic form indicates also the importance of quantum theory on Riemannian (Euclidean) manifolds. From now on we use notation $\mathcal{M}_E$ for these geometries. Euclidean QFT has mathematical advantages, which can be explained by using the example of a free scalar field ϕ with equation

为给出这个平衡态的定义，我们利用“黑洞与欧几里得理论”一节的结论。定义在引力瞬子上的经典引力具有热力学形式, 这也说明了黎曼 (欧几里得) 流形上量子理论的重要性。下文我们用 $\mathcal{M}_E$ 标记这些几何。欧几里得量子场论具有数学上的优势，这可以通过满足如下方程的自由标量场 ϕ 例子来说明

$$P_E\phi = \left(-\nabla^2 + m^2\right)\phi = 0 \tag{71}$$

on $\mathcal{M}_E$. Operator P_E is of a Laplace type. Since $\mathcal{M}_E$ is a curved manifold plane, waves e^{ikx} are not eigen-functions of P_E . However one can act by P_E on a plane wave to get

定义在 $\mathcal{M}_E$ 上。算符 P_E 是拉普拉斯型算符。由于 $\mathcal{M}_E$ 是弯曲流形平面，平面波 e^{ikx} 不是 P_E 的本征函数。不过我们可以用 P_E 作用于平面波得到

$$P_E e^{ik_\mu x^\mu} = \left[k_\mu k_\upsilon g_E^{\mu\upsilon}(x) + \ldots\right] e^{ik_\mu x^\mu}. \tag{72}$$

In mathematical applications, like the spectral theory, the important property of P_E is that its leading symbol $\sigma_P(x,k) = k_\mu k_\upsilon g_E^{\mu\upsilon}(x)$ is not degenerate and is positive-definite on $\mathcal{M}_E$ (so P_E are called elliptic operators). This behavior of the leading symbols is crucially different for operators on Euclidean and Lorentzian manifolds. For elliptic operators, spectral functions, which serve to define other ingredients of the quantum theory, such as the effective action, can be introduced with mathematically meaningful prescriptions since large k^2 asymptotics are under control. More on this topic can be found in monograph [41].

在谱理论这类数学应用中，P_E 的重要性质是其主导符号 $\sigma_P(x,k) = k_\mu k_\upsilon g_E^{\mu\upsilon}(x)$ 非退化，且在 $\mathcal{M}_E$ 上正定 (因此 P_E 被称为椭圆算符)。主导符号的这一性质，在欧几里得流形算符和洛伦兹流形算符之间存在本质区别。对于椭圆算符，由于大 k^2 渐近行为可控，用于定义量子理论其他要素 (比如有效作用量) 的谱函数可以通过数学上有意义的规则引入。更多相关内容可参见专著 [41]。

One of the key quantities which can be rigorously defined is the heat kernel $K(x, y \mid t)$ of P_E which is the solution to the following problem:

热核 $K(x, y \mid t)$ 是可以被严格定义的关键物理量之一，它是 P_E 的热核，满足下述问题的解:

$$\left(\partial_t + P_E(x)\right) K(x, y \mid t) = 0,\ K(x, y \mid t)_{t\to 0} = \delta(x, y). \tag{73}$$

Here t is a positive parameter, $P_E(x)$ acts on argument $x, \delta(x,y) = \delta^4(x-y)/\sqrt{g}$, and the symmetry, $K(x, y \mid t) = K(y, x \mid t)$, is implied. Suppose that $P_E(x)$ does not have zero eigenvalues. Then, by using the heat kernel, one can define the Green function $G(x, y)$ of the operator

> 其中 t 是正参数，$P_E(x)$ 作用于自变量 $x, \delta(x,y) = \delta^4(x-y)/\sqrt{g}$ ，且满足对称性 $K(x, y \mid t) = K(y, x \mid t)$。假设 $P_E(x)$ 没有零本征值，那么利用热核可以定义该算符的格林函数 $G(x, y)$

$$P_E(x)\, G_E(x, y) = \delta(x, y). \tag{74}$$

One can check with the help of (73) that

> 我们可以借助 (73) 验证得到

$$G_E(x, y) = \int_0^\infty dt K(x, y \mid t). \tag{75}$$

Hartle and Hawking [43] used extension of $G_E(x, y)$, via the inverse Wick rotation to the corresponding Lorentzian space-time $\mathcal{M}$, to define Green’s function $G(x, y)$ on $\mathcal{M}$. It is a unique quantum state fixed in this way that is called the HHI state.

> Hartle 与 Hawking[43] 通过将 $G_E(x, y)$ 延拓，再对对应的洛伦兹时空 $\mathcal{M}$ 做逆威克转动，在 $\mathcal{M}$ 上定义了格林函数 $G(x, y)$。通过这种方式固定得到的唯一量子态就被称为 HHI 态。

Below we briefly describe statistical-mechanical interpretation of Euclidean QFT’s for quantum fields on stationary space-times. Let $\mathcal{M}$ be a stationary spacetime with the time-like Killing field $\zeta = \partial_t$. Let $g_{\mu\nu}$ be components of the metric of $\mathcal{M}$ in coordinates $x^\mu(t) = \left(t, x^i\right)$. First suppose that this coordinate chart covers $\mathcal{M}$ globally, that is, $\mathcal{M}$ has the structure $\sum \times R^1$, where $\sum$ are constant t hypersurfaces. Consider decomposition (31) of the field $\phi(x)$ on modes which are eigen-functions of the operator $\zeta = \partial_t$, see (33). Since $\mathcal{M}$ is stationary, one can define a finite-temperature state of ϕ at temperature $T = \beta^{-1}$. The average of an operator $\hat{\mathcal{O}}$ in this state is

> 下文我们简要介绍稳态时空中量子场的欧几里得量子场论的统计力学诠释。设 $\mathcal{M}$ 是具有类光 Killing 场 $\zeta = \partial_t$ 的稳态时空，$g_{\mu\nu}$ 是 $\mathcal{M}$ 的度规在坐标 $x^\mu(t) = \left(t, x^i\right)$ 下的分量。首先假设该坐标图全局覆盖 $\mathcal{M}$，即 $\mathcal{M}$ 具有结构 $\sum \times R^1$，其中 $\sum$ 是恒定 t 超曲面。考虑将场 $\phi(x)$ 分解为算符 $\zeta = \partial_t$ 的特征函数模，参见 (33)。由于 $\mathcal{M}$ 是稳态的，我们可以定义 ϕ 在温度 $T = \beta^{-1}$ 下的有限温度态。该态中算符 $\hat{\mathcal{O}}$ 的平均值为

$$\langle \hat{\mathcal{O}} \rangle_\beta = Z^{-1}(\beta)\, \mathrm{Tr}\left(\hat{\mathcal{O}} e^{-\beta :\hat{H}:}\right), \tag{76}$$

where $Z(\beta)$ is the partition function (58) and the Hamiltonian $:\hat{H}:$ generates evolution along t . One is usually interested in a relation between the Wightman functions

> 其中 $Z(\beta)$ 是配分函数 (58)，哈密顿量: $\hat{H}$: 生成沿 t 的演化。人们通常关注怀特曼函数之间的关系

$$G^+_\beta(x,x') = \left\langle \hat{\phi}(x)\hat{\phi}^+(x')\right\rangle_\beta,\ G^-_\beta(x,y) = \left\langle \hat{\phi}^+(x')\hat{\phi}(x)\right\rangle_\beta \tag{77}$$

on $\mathcal{M}$ and Green's function $G_E(x,y)$ on $\mathcal{M}_E$. It is assumed that $\mathcal{M}$ and $\mathcal{M}_E$ are connected via the Wick rotation $t \to -i\tau$. The inverse components of the Lorentzian and Euclidean metrics are $g_E^{\tau\tau} = -g^{tt}, g_E^{\tau k} = ig^{tk}, g_E^{ik} = g^{ik}$. In general, $\mathcal{M}_E$ is a complex manifold, and $P_E(x)$ is not self-adjoint, but it is still elliptic, which is enough to define with its help corresponding spectral functions and the effective action.

即 $\mathcal{M}$ 上的怀特曼函数与 $\mathcal{M}_E$ 上的格林函数 $G_E(x,y)$。一般认为 $\mathcal{M}$ 与 $\mathcal{M}_E$ 通过威克转动 $t \to -i\tau$ 相联系。洛伦兹度规和欧几里得度规的逆分量为 $g_E^{\tau\tau} = -g^{tt}, g_E^{\tau k} = ig^{tk}, g_E^{ik} = g^{ik}$ 。一般而言，$\mathcal{M}_E$ 是复流形，且 $P_E(x)$ 不是自伴算符，但它仍是椭圆算符，这足以借助它定义对应的谱函数和有效作用量。

Let $x(t)$ be points on an integral line of ∂_t on $\mathcal{M}$ and $\bar{x}(\tau) = x(-i\tau)$ be points on the corresponding integral line of ∂_τ on $\mathcal{M}_E$. One can define the two-point function

设 $x(t)$ 是 $\mathcal{M}$ 上 ∂_t 积分曲线上的点，$\bar{x}(\tau) = x(-i\tau)$ 是 $\mathcal{M}_E$ 上对应 ∂_τ 积分曲线上的点，我们可以定义两点函数

$$\widetilde{G}_\beta(x(z), x'(0)) = \theta(-\Im z)\, G^+_\beta(x(z), x'(0)) + \theta(\Im z)\, G^-_\beta(x(z), x'(0)), \tag{78}$$

where $z = t + i\tau$. It can be shown, see, e.g.,[41], that (78) is an analytic function of z everywhere in the strip $-\beta < \Im z < \beta$ except the domains where the Wightman functions have singularities, that the periodicity property, $\widetilde{G}_\beta(x(z-i\beta), x'(0)) = \widetilde{G}_\beta(x(z), x'(0))$, holds, and that there is the fundamental relation

其中 $z = t + i\tau$。可以证明，例如参见文献 [41]，(78) 是 z 在带域 $-\beta < \Im z < \beta$ 内除去怀特曼函数奇点区域外的处处解析函数，周期性性质 $\widetilde{G}_\beta(x(z-i\beta), x'(0)) = \widetilde{G}_\beta(x(z), x'(0))$ 成立，且存在基本关系

$$G_E(\bar{x}(\tau), x(0)) = \widetilde{G}_\beta(x(-i\tau), x'(0)), \tag{79}$$

between Euclidean and thermal Green's functions.

该关系连接了欧几里得格林函数与热格林函数。

In case of black hole coordinates $x^\mu(t) = (t, x^i)$ cover only a part of $\mathcal{M}$ located to the right from the horizon $\mathcal{H}^+ \cup \mathcal{H}^-$, see Fig. 1. Constant time hypersurfaces $\sum$ intersect on $\mathcal{B}$. The analysis [43] shows that if the Euclidean Green function in (79) is defined on a black hole instanton, at $\beta = \beta_H = 1/T_H$, the corresponding finite-temperature Green function can be extended to the entire black hole space-time $\mathcal{M}$, beyond the domain of stationary coordinates. The quantum state defined by such a Green function is the Hartle-Hawking-Israel state. (When applying this analysis to Kerr black holes, one should note that ζ is time-like in a restricted domain and that non-diagonal components of $\mathcal{M}_E$ can be made real by going to imaginary values of the angular momentum, see section "Black Holes and Euclidean Theory".) If β is an arbitrary parameter, there are some peculiarities in the behavior of Green's functions due to conical singularities discussed in next sections.

在黑洞坐标的情况下，$x^\mu(t) = (t, x^i)$ 仅覆盖位于视界 $\mathcal{H}^+ \cup \mathcal{H}^-$ 右侧的 $\mathcal{M}$ 的一部分，参见图 1。等时超曲面 $\sum$ 在 $\mathcal{B}$ 相交。文献 [43] 的分析表明，若 (79) 中的欧几里得格林函数定义在黑洞瞬子上，在 $\beta = \beta_H = 1/T_H$ 处，对应的有限温度格林函数可延拓到整个黑洞时空 $\mathcal{M}$ ，超出稳态坐标的定义域。由这种格林函数定义的量子态就是哈特尔-霍金-以色列态。(将该分析应用于克尔黑洞时，需要注意 ζ 在受限区域内是类时的，且通过取角动量的虚数值可使 $\mathcal{M}_E$ 的非对角分量变为实数，参见“黑洞与欧几里得理论”章节。) 若 β 为任意参数，格林函数的行为会因后面章节讨论的锥形奇点存在一些特殊性质。

By the construction, a Killing observer sees the HHI state as a thermal bath at the Hawking temperature T_H . In the near-horizon approximation (39),(40), the single-particle excitations for such observers are defined by the set of left-moving and right-moving modes

根据构造，基林观测者将 HHI 态视为霍金温度 T_H 下的热浴。在近视界近似 (39)、(40) 下，这类观测者的单粒子激发由一组左行模和右行模定义

$$f^{(+)}_{L,\omega}(v) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v}, \ f^{(+)}_{R,\omega}(u) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u}, \tag{80}$$

which are eigen-functions of $\zeta = \partial_t$. In case of an eternal Schwarzschild black hole, modes $f^{(+)}_{L,\omega}$ start from $\mathcal{I}^-$ and end on the black hole singularity, while $f^{(+)}_{R,\omega}$ start at the white hole singularity and move to $\mathcal{I}^+$, see Fig. 1.

它们是 $\zeta = \partial_t$ 的本征函数。对于永恒史瓦西黑洞，模 $f^{(+)}_{L,\omega}$ 从 $\mathcal{I}^-$ 出发终止于黑洞奇点，而 $f^{(+)}_{R,\omega}$ 从白洞奇点出发运动到 $\mathcal{I}^+$ ，参见图 1。

One can also consider modes

也可以考虑这些模

$$\bar{f}^{(+)}_{L,\omega}(v) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega V(v)}, \ \bar{f}^{(+)}_{R,\omega}(u) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega U(u)}, \tag{81}$$

where V and U are Kruskal coordinates (45). The HHI state is the vacuum state for quasiparticles associated with modes (81). To check this one can calculate the number of particles related to (80) in a vacuum state for (81) by using (37). The relevant nontrivial Bogoliubov coefficients between R -modes are

其中 V 和 U 是克鲁斯卡尔坐标 (45)。HHI 态是与 (81) 式模相关的准粒子的真空态。可以通过 (37) 计算 (81) 式真空态中与 (80) 式相关的粒子数来验证这一点。R 模之间相关的非平庸博戈留波夫系数为

$$d^R_{\sigma,\omega} = \left\langle f^{(+)}_{R,\sigma}, \bar{f}^{(-)}_{R,\omega} \right\rangle = -\frac{1}{2\pi}\sqrt{\frac{\sigma}{\omega}} \int_{-\infty}^{\infty} du e^{i\sigma u + i\omega U(u)}. \tag{82}$$

As follows from (51), the integral in the r.h.s. of (82) is given by (52). Thus (82) corresponds to the thermal distribution of quanta at the Hawking temperature, which is the property of the HHI state. The corresponding coefficient between the left modes has analogous expression.

由 (51) 可知，(82) 右侧的积分由 (52) 给出。因此 (82) 对应霍金温度下量子的热分布，这正是 HHI 态的性质。左模之间的对应系数有类似的表达式。

The HHI state can be defined on a global Cauchy surface where it has an important representation. Consider, as an example, the Carter-Penrose diagram of an eternal Schwarzschild black hole. A constant time hypersurface $\sum_R \cup \sum_L$ which goes from the right to the left world is called the Einstein-Rosen bridge, see Fig. 1. The left and right parts of the bridge are identical and coincide with constant time sections. In HHI wave function, in the configuration representation, depends on field variables φ_R, φ_L set on $\sum_R, \sum_L$, correspondingly. The wave function can be represented as transition amplitude in the Euclidean time τ between the left and right worlds

HHI 态可以定义在整体柯西曲面上，在该曲面上它有一个重要表示。举例来说，考虑永恒史瓦西黑洞的卡特-彭罗斯图。从右侧世界延伸到左侧世界的等时超曲面 $\sum_R \cup \sum_L$ 称为爱因斯坦-罗森桥，参见图 1。桥的左右部分完全相同，且与等时截面重合。HHI 波函数在组态表象中依赖于分别设定在 $\sum_R, \sum_L$ 上的场变量 φ_R, φ_L 。该波函数可以表示为左右世界之间欧几里得时间 τ 下的跃迁振幅

$$\langle \varphi_L, \varphi_R \mid HHI \rangle = N^{-1/2} \left\langle \varphi_L \middle| e^{-\frac{\beta_H}{2}:\hat{H}:} \middle| \varphi_R \right\rangle = \int [D\phi]\, e^{-I_E[\varphi_L, \varphi_R]}, \tag{83}$$

where N is a normalization factor, and the Euclidean action,

其中 N 是归一化因子，欧几里得作用量

$$I_E[\varphi_L, \varphi_R] = -\frac{1}{2}\int_0^{\beta_H/2} d\tau \int d^3x \phi^* P_E \phi, \tag{84}$$

is defined with boundary conditions $\phi(\tau = 0, x^i) = \varphi_R(x^i), \phi(\tau = \beta_H/2, x^i) = \varphi_L(x^i)$. A detailed discussion of this representation, which is a natural generalization of a similar formula for the Minkowsky vacuum [52], can be found in [5].

由边界条件 $\phi(\tau = 0, x^i) = \varphi_R(x^i), \phi(\tau = \beta_H/2, x^i) = \varphi_L(x^i)$ 定义。该表示是闵氏真空类似公式的自然推广 [52]，详细讨论可以参见文献 [5]。

Effective Action and Renormalized Stress-Energy Tensor

有效作用量与重整化能量动量张量

Vacuum polarization in an external gravitational field $g_{\mu\nu}$ results a nontrivial average of the stress-energy tensor of a quantum field, $\langle \hat{T}_{\mu\nu} \rangle$, which appears in the right-hand side of the Einstein equations (1). The advantage of the Euclidean formulation of the theory is that $\langle \hat{T}_{\mu\nu} \rangle$ in the HHI state can be derived by variation of the effective action.

外引力场中的真空极化 $g_{\mu\nu}$ 会得到量子场能量动量张量的非平庸平均值 $\langle\hat{T}_{\mu\nu}\rangle$，该平均值出现在爱因斯坦方程 (1) 的右侧。该理论欧几里得表述的优势在于，HHI 态下的能量动量张量平均值可以通过对有效作用量变分得到 $\langle\hat{T}_{\mu\nu}\rangle$。

An introduction to the effective action approach in perturbative quantum gravity can be found in [14]. The effective action of noninteracting quantum fields can be defined as the following functional on $\mathcal{M}_E$:

微扰量子引力中有效作用量方法的入门介绍可参见文献 [14]。自由量子场的有效作用量可以定义为作用于 $\mathcal{M}_E$ 上的如下泛函:

$$\Gamma[g] = I\left[g, \left(G^B, \Lambda^B, c_i^B\right)\right] + W[g]. \tag{85}$$

Here $I[g]$ is the gravity action (67) and G^B, Λ^B, c_i^B are bare couplings. The quantum part of the action is

此处 $I[g]$ 是引力作用量 (67)，G^B, Λ^B, c_i^B 是裸耦合常数。作用量的量子部分为

$$W[g] = \frac{\eta}{2}\ln\det P_E, \tag{86}$$

where $\eta = +1$ or -1 for fields with Bose or Fermi statistics, respectively. Expression (86) is motivated by a formal path integral for free fields. It needs a further prescription to deal with ultraviolet divergences. Since P_E is an elliptic operator, one can introduce the heat trace (see details in [41])

其中对于玻色统计场和费米统计场，$\eta = +1$ 分别为 +1 或-1。表达式 (86) 由自由场的形式路径积分导出，它需要额外的处理方案来解决紫外发散问题。由于 P_E 是椭圆算子，我们可以引入热迹 (详见文献 [41])

$$K(P_E;t) = \sum_\lambda e^{-t\lambda} \tag{87}$$

where the sum is taken over all eigenvalues λ of P_E . If λ are positive, one can use the definition:

其中求和对 P_E 的所有本征值 λ 进行。若 λ 为正，我们可以使用如下定义:

$$\ln\det P_E = -\int_\delta^\infty \frac{dt}{t} K(P_E;t), \tag{88}$$

where $\delta > 0$ is a proper cutoff parameter. An important property of (87) is a short t expansion (as $t \to +0$)

其中 $\delta > 0$ 是合适的截断参数。式 (87) 的一个重要性质是短 t 展开 (当 $t \to +0$ 时)

$$K(P_E;t) \sim \sum_{p=0}^\infty t^{\frac{p-n}{2}} a_p(P_E), \tag{89}$$

where n is the dimensionality of $\mathcal{M}_E$, and $a_p(P_E)$ with odd p appear if $\mathcal{M}_E$ has boundaries. The heat kernel (or DeWitt-Seeley) coefficients $a_p(P_E)$ determine the divergent part of the effective action

其中 n 是 $\mathcal{M}_E$ 的维度，当 $\mathcal{M}_E$ 存在边界时会出现带有奇数 p 的 $a_p(P_E)$ 项。热核 (即德维特-西利) 系数 $a_p(P_E)$ 决定了有效作用量的发散部分

$$W_{\text{div}}[g,\delta]=\eta\sum_{p=0}^{n-1}\frac{a_p(P_E)}{p-n}\delta^{\frac{p-n}{2}}+\eta a_n(P_E)\ln\delta, \tag{90}$$

where we used (88). A common prescription to eliminate the ultraviolet divergences is to note that $a_p(P_E)$ with $p=2k$ are integrals of the k -th order polynomials in the curvature tensor. Thus, in four dimensions, (90) has the same structure as the bare action $I\left[g,\left(G^B,\Lambda^B,c_i^B\right)\right]$. The divergences are eliminated by redefinition of G^B,Λ^B,c_i^B ,

此处我们使用了式 (88)。消除紫外发散的常用方法注意到，带有 $p=2k$ 的 $a_p(P_E)$ 是曲率张量上 k 次多项式的积分。因此在四维中，式 (90) 与裸作用量 $I\left[g,\left(G^B,\Lambda^B,c_i^B\right)\right]$ 结构相同。发散可以通过重新定义 G^B,Λ^B,c_i^B 消除,

$$I\left[g,(G,\Lambda,c_i)\right]=I\left[g,\left(G^B,\Lambda^B,c_i^B\right)\right]+W_{\text{div}}[g,\delta]. \tag{91}$$

This yields effective action (85) in the renormalized form:

这就得到了重整化形式下的有效作用量 (85):

$$\Gamma[g]=I\left[g,(G,\Lambda,c_i)\right]+W_{\text{ren}}[g], \tag{92}$$

where renormalized or the UV-finite part is

其中重整化的紫外有限部分为

$$W_{\text{ren}}[g]=\lim_{\delta\to 0}\left(W[g,\delta]-W_{\text{div}}[g,\delta]\right). \tag{93}$$

Formula (92) can be used to derive the gravity equations which allow one to take into account the back-reaction to quantum fields

公式 (92) 可用于推导引力方程，从而将量子场的反作用考虑进来

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\ldots=\frac{8\pi G}{c^4}T^E_{\mu\nu}, \tag{94}$$

$$T^{E\mu\upsilon}[g]=\frac{2}{\sqrt{\det|g|}}\frac{\delta W_{\text{ren}}[g]}{\delta g_{\mu\upsilon}}. \tag{95}$$

The dotes in (94) are quadratic in curvature terms. To calculate the first quantum correction to the metric tensor $g_{\mu\upsilon}$, it is enough to consider $T^{E\mu\upsilon}[g]$ on the corresponding classical black hole instanton.

式 (94) 中的点代表曲率项的二次项。要计算度规张量 $g_{\mu\upsilon}$ 的一阶量子修正，只需在对应的经典黑洞瞬子上考虑 $T^{E\mu\upsilon}[g]$ 即可。

Several remarks are in order.

这里做几点说明。

(i) One can analytically continue (94) from the Euclidean to the Lorentzian theory. If we are interested in the first quantum correction, the Lorenzian theory is just the gravity theory sourced by the average of the stress-energy tensor in the Hartle-Hawking-Israel state. On black hole instanton,

(i) 我们可以将式 (94) 从欧几里得理论解析延拓到洛伦兹理论。如果我们关注一阶量子修正，那么洛伦兹理论就是以哈特尔-霍金-以色列态下能量动量张量平均值为源的引力理论。在黑洞瞬子上，

$$T^E_{\mu\nu}[g] = i^q \langle HHI \left| \hat{T}_{\mu\nu}[g] \right| HHI \rangle, \tag{96}$$

where q is the number of temporal indexes and the factor i^q is related to the Wick rotation. Given connection (79) between the Euclidean and finite-temperature Green's functions relation (96) can be proved when the quantum stress-energy tensor is computed with the help of a Green's function by using well-known point-splitting procedure. (The point-splitting method is discussed in detail in [11].)

其中 q 是时间指标的数目，因子 i^q 与威克转动相关。已知欧几里得格林函数与有限温格林函数之间存在联系 (79)，当我们利用格林函数、通过常用的点分裂方案计算量子能量动量张量时，就可以证明关系式 (96)。(点分裂方法的详细讨论参见文献 [11]。)

(ii) By the construction, (96) yields the stress-energy tensor in stationary coordinates outside the horizon. Since Green's function can be analytically extended to the entire space-time, so does (96).

(ii) 根据构造，式 (96) 给出了视界外稳态坐标中的能量动量张量。由于格林函数可以解析延拓到整个时空，式 (96) 也同样可以。

(iii) In addition to its transparent physical meaning, the HHI state is distinguished by mathematical properties when computations of quantum averages are reduced to finding various spectral functions of elliptic operators P_E .

(iii) 除了物理意义清晰明确之外，HHI 态的特殊之处还在于其数学性质: 计算量子平均值时可简化为求解椭圆算子的各种谱函数 P_E 。

(iv) There are different types of UV regularizations in the effective action, for example, the ζ -function regularization [26, 27, 45] , the dimensional regularization, and the Pauli-Villars regularization [11]. Up to several finite counterterms, all of them yield the same W_{ren} .

(iv) 有效作用量中有多种不同的 UV 正则化方法，例如 ζ 函数正则化 [26, 27, 45] 、维数正则化以及泡利-维拉斯正则化 [11]。除了数个有限抵消项之外，所有这些方法得到的 W_{ren} 都是相同的。

Fluid Dynamics in Gravitational Fields

引力场中的流体动力学

Finite-Temperature QFT's and Effective Action

有限温度量子场论与有效作用量

The connection between the Euclidean and finite-temperature Green's functions suggests that there may exist an analogous relation between the Euclidean effective action W and the free energy of the quantum field. Let us start with stationary space-times of the structure $\sum \times R^1$, where $\sum$ are constant time sections, and assume a time-like Killing field acts globally on $\mathcal{M}$.

欧几里得格林函数与有限温度格林函数之间的关联表明，欧几里得有效作用量 W 与量子场的自由能之间可能存在类似关系。我们从结构为 $\sum \times R^1$ 的稳态时空开始讨论，其中 $\sum$ 是恒定时间截面，并且假设类时基灵场全局作用于 $\mathcal{M}$ 。

For simplicity, we proceed with a free scalar field. Generalization to other fields is possible but requires additional irrelevant details. The wave equation (23) on a stationary space-time can be written as

为简化起见，我们以自由标量场为例进行研究。该结论可以推广到其他场，但需要补充无关的额外细节。稳态时空中的波动方程 (23) 可以写为

$$P\left(\partial_t, \partial_i\right)\phi = 0. \tag{97}$$

The relation between Lorentzian and Euclidean operators is $P_E\left(\partial_\tau, \partial_i\right) = P\left(i\partial_\tau, \partial_i\right)$. To define statistical-mechanical quantities, one needs the single-particle energies ω_i introduced in (33). Their spectrum is determined by the problem

洛伦兹算符与欧几里得算符的关系为 $P_E\left(\partial_\tau, \partial_i\right) = P\left(i\partial_\tau, \partial_i\right)$ 。要定义统计力学量，需要用到 (33) 中引入的单粒子能量 ω_i ，其能谱由下述问题确定

$$\left(P_0\omega^2 + P_1\omega + P_2\right)\phi_\omega = 0, \tag{98}$$

which follows after substitution $\phi\left(t, x^i\right) = e^{-i\omega t}\phi_\omega\left(x^i\right)$ to (97). Here P_k are k - th order partial differential operators. Equation (98) is called a nonlinear spectral problem.

该问题可由将 $\phi\left(t, x^i\right) = e^{-i\omega t}\phi_\omega\left(x^i\right)$ 代入 (97) 得到。此处 P_k 是 k 阶偏微分算子。方程 (98) 称为非线性谱问题。

With the help of the single-particle spectrum, the free energy of the considered system can be written as

借助单粒子能谱，所研究系统的自由能可以写为

$$F(\beta) = -\beta^{-1} \ln Z(\beta) = \beta^{-1} \sum_i \ln\left(1 - e^{-\beta\omega_i}\right), \tag{99}$$

where $Z(\beta)$ is given by (58). The summation goes over all single-particle energies $\omega_i^{\pm}$. For continuous spectra, the sum is replaced with the corresponding integrals.

其中 $Z(\beta)$ 由 (58) 给出。求和遍及所有单粒子能量 $\omega_i^{\pm}$。对于连续能谱，求和需替换为对应的积分。

One can formally define the vacuum energy $E_0 = \frac{1}{2}\sum_i \omega_i$, where the series diverges and should be considered in the framework of some regularization prescription. Let $E_{0,\,\mathrm{ren}}$ be a finite (renormalized) part of E_0 left after subtracting divergent terms. Then there is the relation between the free energy and Euclidean action (86):

我们可以形式化定义真空能量 $E_0 = \frac{1}{2}\sum_i \omega_i$，该级数发散，需要在某种正则化方案下进行处理。设 $E_{0,\,\mathrm{ren}}$ 是减去发散项后 E_0 的有限 (重整化) 部分，则自由能与欧几里得作用量 (86) 存在如下关系:

$$W_{\mathrm{ren}}[g] = \beta\left(F(\beta) + E_{0,\,\mathrm{ren}}\right). \tag{100}$$

The left and right parts of (100) coincide up to finite counterterms. The derivation of (100) can be found, e.g., in [41].

(100) 的左右两侧在有限抵消项范围内一致。例如，(100) 的推导可以在文献 [41] 中找到。

On the considered class of geometries, $F(\beta)$ is finite at large ω_i. As follows from (100), the UV divergent part of the effective action appears from the divergences of the vacuum energy. As we see in section ”Generalized Black Hole Entropy and Its Interpretations”, the situation changes when the Killing field ζ has the horizon.

在我们研究的这类几何中，$F(\beta)$ 在大 ω_i 处有限。由 (100) 可知，有效作用量的紫外发散部分来自真空能量的发散。正如我们在“广义黑洞熵及其诠释”一节中看到的，当基灵场 ζ 存在视界时，情况会发生改变。

High-Temperature Asymptotic

高温渐近

A Killing observer with 4-velocity along $\zeta = \partial_t$ measures the so-called local Tolman temperature

沿 $\zeta = \partial_t$ 方向具有 4-速度的基灵观测者会测量到所谓的局域托尔曼温度

$$T(x) = \frac{T_0}{\sqrt{-\zeta^2}}, \tag{101}$$

where $T_0 = \beta^{-1}$. In asymptotically flat space-times, T_0 is a temperature measured by observers at infinity. Factor $\sqrt{-\zeta^2}$ relates coordinate time t to the proper time of the observer. The Tolman temperature increases if the position of the observer is taken closer to the black hole horizon. One says that temperature is blue-shifted near $\mathcal{H}$.

其中 $T_0 = \beta^{-1}$ 。在渐近平直时空中，T_0 是无穷远观测者测量到的温度。因子 $\sqrt{-\zeta^2}$ 将坐标时间 t 与观测者的固有时联系起来。当观测者的位置越靠近黑洞视界，托尔曼温度就越高，也就是说温度在 $\mathcal{H}$ 附近发生蓝移。

Therefore in the near-horizon region the system is effectively at a high-temperature regime. Interestingly, the high-temperature limit allows an analytic form of the free energy which, in general, can be written in terms of characteristics of the Killing frame of reference, discussed in section "Necessary Definitions". For example, in the absence of boundaries the leading terms look as follows:

因此，在近视界区域，系统实际上处于高温区域。有趣的是，高温极限下自由能可以得到解析形式，一般而言该解析形式可以用基灵参考系的特征量表示，我们已在“必要定义”一节讨论过相关内容。例如，无边界时领头项的形式如下:

$$F(\beta) \simeq -\int \sqrt{-g} d^3 x \left[b_1 T^4(x) + \right.$$

$$\left. T^2(x) \left(b_2 R + b_3 m^2 + b_4 \Omega^2(x) + b_5 w_\mu w^\mu + b_6 \nabla w \right) + O(\ln \beta) \right]. \tag{102}$$

Here w_μ is 4-acceleration, $\Omega = \frac{1}{2}\left(A_{\mu\nu}A^{\mu\nu}\right)^{1/2}$ is the absolute value of the local angular velocity, and $A_{\mu\nu}$ is the rotation tensor, see definitions (14). The structure of (102) is determined by using canonical mass dimensions of all quantities, and b_k are numerical coefficients which depend on the considered model and require computations. For a real scalar field with equation $\left(-\nabla^2 + V\right)\phi = 0$, one finds [37]

此处 w_μ 是 4-加速度，$\Omega = \frac{1}{2}\left(A_{\mu\nu}A^{\mu\nu}\right)^{1/2}$ 是局域角速度的绝对值，$A_{\mu\nu}$ 是转动张量，参见定义 (14)。式 (102) 的结构由所有量的正则质量量纲确定，b_k 是依赖于所研究模型的数值系数，需要通过计算得到。对于满足方程 $\left(-\nabla^2 + V\right)\phi = 0$ 的实标量场，可得 [37]

$$F(\beta) \simeq -\int d^3 x \sqrt{-g} \left[\frac{\pi^2}{90} T^4 + \frac{1}{24} T^2 \left(\frac{1}{6} R - V - \frac{2}{3} \Omega^2 \right) + O(\ln \beta) \right].$$

(103)

According to (100), one can split the stress-energy tensor as

根据式 (100)，可以将能量动量张量分解为

$$\left\langle HHI \left| \hat{T}_{\mu\nu} \right| HHI \right\rangle = \left\langle \hat{T}_{\mu\nu} \right\rangle_\beta + \left\langle \hat{T}_{\mu\nu} \right\rangle_{\text{vac}}. \tag{104}$$

The thermal part $\langle T_{\mu\nu} \rangle_\beta$ is determined by the free energy $F(\beta)$, while the vacuum part $\langle T_{\mu\nu} \rangle_{\text{vac}}$ follows from the vacuum energy. Variations of (103) over the metric yield the thermal part at high temperatures. For example, its expression in the case of conformal coupling, $V = R/6$, is [37]

热部分 $\langle T_{\mu\nu} \rangle_\beta$ 由自由能 $F(\beta)$ 确定，而真空部分 $\langle T_{\mu\nu} \rangle_{\text{vac}}$ 由真空能量得到。对式 (103) 关于度规变分可得到高温下的热部分。例如，共形耦合 $V = R/6$ 情况下的表达式为 [37]

$$\langle T_{\mu\nu}\rangle_\beta \sim (g_{\mu\nu} + 4u_\mu u_\nu)\left(\frac{\pi^2}{90}T^4 - \frac{1}{36}T^2\Omega^2\right)$$

$$+\frac{T^2}{36}\left(2A_{\nu\rho}A_\mu{}^\rho - u_\nu A_{\mu\lambda}w^\lambda - u_\mu A_{\nu\lambda}w^\lambda + u_\nu A_{\mu\lambda}{}^{;\lambda} + u_\mu A_{\nu\lambda}{}^{;\lambda}\right) + O(\ln\beta)\,.$$

(105)

One can check that $\nabla^\mu\langle T_{\mu\nu}\rangle_\beta = 0, g^{\mu\nu}\langle T_{\mu\nu}\rangle_\beta = 0$. High-temperature asymptotics in static space-times have been studied in pioneering papers [24,25]. The extension to stationary geometries, which requires one to deal with nonlinear spectral problems (98), has been suggested in [37], see [41] for more details.

> 可以验证 $\nabla^\mu\langle T_{\mu\nu}\rangle_\beta = 0, g^{\mu\nu}\langle T_{\mu\nu}\rangle_\beta = 0$ 。静态时空的高温渐近已在开创性文献 [24,25] 中得到研究。对稳态几何的推广需要处理非线性谱问题 (98)，该推广由文献 [37] 提出，更多细节参见 [41]。

Asymptotics (102) and (103) are of increasing interest in modern studies of equilibrium distributions in finite-temperature quantum field theories with rotation and acceleration in flat space-times [7], where $A_{\mu\nu}$ is interpreted as a thermal vorticity. The interest is related to properties of quark-gluon plasma in heavy ion collisions.

> 渐近式 (102) 和 (103) 在当前研究中越来越受关注，这些研究围绕平直时空中存在转动和加速度的有限温度量子场论的平衡分布展开 [7]，其中 $A_{\mu\nu}$ 被解释为热涡度。这种关注度与重离子碰撞中夸克-胶子等离子体的性质有关。

Generalized Thermodynamics of Quantum Black Holes

> ## 量子黑洞的广义热力学

Generalized Free Energy

> ## 广义自由能

The HHI state describes an eternal black hole in thermal equilibrium with its radiation. The system, a black hole and quantum fields in the HHI state, can be called a quantum black hole. The split of this system on classical and quantum components is rather conditional. The classical black hole geometry back-reacts to quantum matter, while bare classical couplings participate in renormalization of ultraviolet divergences.

> HHI 态描述了与自身辐射处于热平衡的永恒黑洞。该系统 (处于 HHI 态的黑洞与量子场) 可称为量子黑洞。将该系统拆分为经典组分和量子组分是相当条件性的。经典黑洞几何会对量子物质产生反作用，而裸经典耦合会参与紫外发散的重整化。

Black hole thermodynamics poses a number of fundamental questions to quantum gravity mentioned in previous sections. As a first step in resolving the existing paradoxes, it is reasonable to understand thermodynamics of quantum black holes in the HHI state. We describe here some steps how it can be done by using advantages of the Euclidean field theory. For simplicity, we consider nonrotating black holes. The extension of the arguments below to black holes with angular momentum is possible, but it does not carry principle issues.

黑洞热力学对前文提及的量子引力提出了若干基础问题。作为解决现有悖论的第一步，合理的方向是理解 HHI 态下量子黑洞的热力学。我们在此说明如何利用欧几里得场论的优势推进这一研究。为简化讨论，我们考虑非旋转黑洞。将下述论证推广至带角动量的黑洞是可行的，且不存在原则性问题。

The key assumption we adopt is that the generalized free energy of a quantum black hole is determined by the effective action $\Gamma[g]$, see (92). The background metric, $\bar{g}$, is a stationary point of $\Gamma[g]$ under certain boundary conditions, which can be defined as in the classical case [83]. One also requires that $\bar{g}$ is a black hole instanton-like solution with a global Euclidean Killing vector field ∂_τ, τ being a periodic coordinate with a period β . We interpret β as an inverse temperature.

我们采用的核心假设是: 量子黑洞的广义自由能由有效作用量 $\Gamma[g]$ 确定，见式 (92)。背景度规 $\bar{g}$ 是 $\Gamma[g]$ 在特定边界条件下的驻点，可按经典情况的方式定义 [83]。还要求 $\bar{g}$ 是类瞬子黑洞解，整体欧几里得基灵矢量场 ∂_τ, τ 为周期坐标，周期为 β 。我们将 β 解释为逆温度。

Variations of $\Gamma[g]$ lead to the Einstein equations (94) sourced by the stress-energy tensor of the quantum matter. The functional $W[g]$ is a complicated nonlocal functional which is known in some approximations, in the second-order curvature approximation [6], for example. Quantum corrected Schwarzschild solutions in the framework of these approximations have been discussed in [16, 82].

对 $\Gamma[g]$ 变分可得到由量子物质能量动量张量作为源的爱因斯坦方程 (94)。泛函 $W[g]$ 是复杂的非局域泛函，仅在部分近似下已知，例如二阶曲率近似 [6]。这些近似框架下量子修正的史瓦西解已在文献 [16, 82] 中讨论。

The generalized free energy of a quantum black hole is defined as follows:

量子黑洞的广义自由能定义如下:

$$F_{\text{gen}}(\beta) = \beta^{-1}\Gamma[\bar{g}(\beta)]. \tag{106}$$

There are some subtle issues related to normalization of $\Gamma[\bar{g}(\beta)]$ in (106) by analogy with the Gibbons-Hawking subtraction procedure [42]. We do not imply any subtraction in the quantum part of the effective action before we find its statistical-mechanical interpretation. Equation (106) allows one to introduce generalized energy and entropy of the black hole by using standard definitions of statistical physics

与 (106) 中 $\Gamma[\bar{g}(\beta)]$ 的归一化相关的一些微妙问题，可以类比吉本斯-霍金减法手续 [42] 得到。在找到有效作用量量子部分的统计力学解释之前，我们不对其做任何减除。式 (106) 允许我们利用统计物理的标准定义引入黑洞的广义能量和熵。

$$E_{\text{gen}}(\beta) = \partial_\beta\left(\beta F_{\text{gen}}(\beta)\right), \; S_{\text{gen}}(\beta) = \beta^2\partial_\beta\left(F_{\text{gen}}(\beta)\right). \tag{107}$$

By virtue of (107), parameters of two black holes with slightly different temperatures are related by the first law

根据 (107)，两个温度相差微小的黑洞的参数满足热力学第一定律:

$$\delta E_{\text{gen}}(\beta) = \beta^{-1}\delta S_{\text{gen}}(\beta), \tag{108}$$

which is the generalization of (56) for $\Omega_H = 0$.

这是 (56) 对 $\Omega_H = 0$ 的推广。

The generalized free energy and entropy can be found in some cases, for example, for a Schwarzschild black hole with massless quantum fields [35, 68]. This can be done by using scaling properties of $W_{\text{ren}}(\beta)$ and leads to logarithmic corrections to the Bekenstein-Hawking entropy. The result can be generalized to include higher loops [72]. Corrections for other types of black holes are discussed in [64,69].

在部分情形下可以求得广义自由能和熵，例如带无质量量子场的史瓦西黑洞 [35, 68]。这可以利用 $W_{\text{ren}}(\beta)$ 的标度性质完成，并且会给贝肯斯坦-霍金熵带来对数修正。该结果可以推广到包含高阶圈的情况 [72]。其他类型黑洞的修正已在文献 [64,69] 中讨论。

To proceed with (107), we note that the derivative of the effective action over β can be taken in two steps

为进一步讨论 (107)，我们注意到有效作用量对 β 的导数可以分两步计算

$$\partial_\beta \Gamma[\bar{g}] = \partial_\beta \Gamma[\bar{g}]_{\bar{g}} + \int d^4x\sqrt{\bar{g}}\frac{\delta\Gamma[\bar{g}]}{\delta g_{\mu\nu}(x)}\partial_\beta \bar{g}_{\mu\nu}(x) + B. \tag{109}$$

The first term, $\partial_\beta\Gamma[\bar{g}]_{\bar{g}}$, is the derivative over β when the bulk metric is fixed. This derivative changes the periodicity of the instanton $\bar{g}$ and results in conical singularities at the Euclidean horizon $\mathcal{B}$. The second and third terms in the r.h.s. of (109) come out when one differentiates the metric but keeps the periodicity fixed. The third term, B , appears from the "inner boundary" $\mathcal{B}$. This term is related to singularities on $\mathcal{B}$. The "inner boundary" $\mathcal{B}$ in the off-shell approach should be taken into account when integrating by parts to get rid of derivatives of $\delta\bar{g}$. Since there are no singularities on the physical boundary $\partial\mathcal{M}_E$, we assume that the corresponding boundary terms in (109) are excluded by appropriate boundary conditions.

第一项 $\partial_\beta\Gamma[\bar{g}]_{\bar{g}}$ 是 bulk 度规固定时对 β 的导数。该导数会改变瞬子 $\bar{g}$ 的周期性，在欧几里得视界 $\mathcal{B}$ 处产生锥奇点。(109) 右侧的第二项和第三项来自固定周期性、对度规求导的情况。第三项 B 来自“内边界” $\mathcal{B}$ ，该项与 $\mathcal{B}$ 上的奇点相关。脱壳方法中，分部积分消去 $\delta\bar{g}$ 的导数时必须考虑“内边界” $\mathcal{B}$ 。由于物理边界 $\partial\mathcal{M}_E$ 上不存在奇点，我们认为 (109) 中对应的边界项已被适当的边界条件排除。

Since $\bar{g}$ is a stationary point of $\Gamma[g]$, the second term in the r.h.s of (109) disappears, and one gets

由于 $\bar{g}$ 是 $\Gamma[g]$ 的驻点，式 (109) 右侧第二项消失，可得

$$E_{\text{gen}}(\beta) = \partial_\beta\Gamma[\bar{g}]_{\bar{g}} + B. \tag{110}$$

$$S_{\text{gen}}(\beta) = \left(\beta\partial_\beta - 1\right)\Gamma[\bar{g}]_{\bar{g}} + \beta B. \tag{111}$$

It follows from (92) and properties of the classical Euclidean action (62) and (107) that

由式 (92) 以及经典欧几里得作用量的性质 (62) 和 (107) 可得

$$E_{\text{gen}}(\beta) = \bar{M}_H(\beta) + E_q(\beta) + B,\ S_{\text{gen}}(\beta) = \bar{S}^{BH}(\beta) + S_q(\beta) + \beta B. \tag{112}$$

Here $\bar{M}_H(\beta)$ and $\bar{S}^{BH}(\beta)$ are the mass and Bekenstein-Hawking entropy (for simplicity, we assume that renormalized couplings $c_i = 0$ in (92)) computed by applying the classical formula to the quantum corrected instanton $\bar{g}$. We recall that β^{-1} is identified with the Hawking temperature. Other quantum corrections in (112) are defined as

此处 $\bar{M}_H(\beta)$ 和 $\bar{S}^{BH}(\beta)$ 分别是将经典公式应用于量子修正瞬子 $\bar{g}$ 计算得到的质量与贝肯斯坦-霍金熵 (为简便起见，我们假设已重整化耦合 $c_i = 0$ 处于式 (92) 中)。我们重申 β^{-1} 对应霍金温度。式 (112) 中的其他量子修正定义为

$$E_q(\beta) = \partial_\beta W_{\text{ren}}(\beta),\ S_q(\beta) = \left(\beta\partial_\beta - 1\right) W_{\text{ren}}(\beta). \tag{113}$$

Since we are interested in first-order quantum corrections, it is enough to take $W_{\text{ren}}(\beta)$ on the classical instanton.

由于我们关注一阶量子修正，因此只需在经典瞬子上取 $W_{\text{ren}}(\beta)$ 即可

Energy of a Quantum Black Hole

量子黑洞的能量

Consider first classical black holes. The total mass M of a black hole, as measured at infinity, is defined by the differential mass formula [4]:

首先考虑经典黑洞。在无穷远处测得的黑洞总质量 M 由微分质量公式定义 [4]:

$$M = M_H + E \tag{114}$$

$$E = -\int_\Sigma d^3x\sqrt{-g}T_0^0, \tag{115}$$

where M_H is the mass measured at the horizon, and E is the energy of matter outside the horizon. The covariant stress-energy tensor of matter T_υ^μ for field theories is determined by the variation of the corresponding action over the metric. The integral in (115) is defined on a constant time slice $\sum$ outside the black hole.

其中 M_H 是视界面处测得的质量，E 是视界外物质的能量。场论中物质的协变应力-能量张量 T_υ^μ 由对应作用量对度规的变分确定。(115) 中的积分定义在黑洞外的等时切片 $\sum$ 上。

Equation (114) suggests the following natural form of the generalized energy of a quantum black hole:

式 (114) 给出了量子黑洞广义能量的如下自然形式:

$$E_{\text{gen}}(\beta) = \bar{M}_H(\beta) + E(\beta), \tag{116}$$

$$E(\beta) = -\int_{\Sigma} d^3x\sqrt{-g}\langle HHI\left|\hat{T}_0^0\right|HHI\rangle. \tag{117}$$

$\bar{M}_H$ is identified with the mass at the horizon, while E is replaced with the expectation value of the corresponding energy operator in the HHI state.

$\bar{M}_H$ 对应视界面处的质量，而 E 替换为对应能量算符在 HHI 态中的期望值。

Consider now quantum correction E_q in (112). This correction is defined in (113). It follows from (100) and thermal properties of the HHI state that

现在考虑 (112) 中的量子修正 E_q，该修正由 (113) 定义。由 (100) 和 HHI 态的热性质可得:

$$E_q(\beta) = \langle :\hat{H}:\rangle_\beta + E_{0,\text{ ren}} = \langle HHI\left|\hat{H}\right|HHI\rangle, \tag{118}$$

The r.h.s. of (118) is the renormalized expectation value of the canonical energy operator $\hat{H}$ without any normal ordering.

(118) 的右侧是正则能量算符 $\hat{H}$ 未经正规序处理的重整化期望值。

The definition of energy by (115) in terms of the metric stress-energy tensor and the definition of energy as the Hamiltonian may differ by a total derivative which results in a nonvanishing term when $\sum$ has $\mathcal{B}$ as an internal boundary, where the Killing field bifurcates. One can show that [28, 36]

用 (115) 通过度规应力-能量张量定义的能量，与将能量定义为哈密顿量，二者相差一个全导数；当 $\sum$ 以 $\mathcal{B}$ 作为内边界、且 Killing 场在该处分叉时，该全导数会产生一个非零项。可以证明 [28, 36]

$$E = H - \beta Q. \tag{119}$$

Here β is the inverse Hawking temperature determined by the surface gravity of ζ, and Q is an integral over $\mathcal{B}$. The quantity Q is always nontrivial when there are non-minimal couplings between dynamical fields and the background curvature. It can be demonstrated for black holes in a general classical theory of gravity arising from a diffeomorphism invariant Lagrangian that [36]:

此处 β 是由 ζ 的表面引力确定的霍金温度倒数，Q 是 $\mathcal{B}$ 上的积分。当动力学场与背景曲率存在非最小耦合时，Q 始终不为零。对于由微分同胚不变拉格朗日量得到的广义经典引力理论中的黑洞，可以证明 [36]:

(i) E appears in the first law of black hole mechanics and plays the same role as the energy in the differential mass formula (114).

(i) E 出现在黑洞力学第一定律中，发挥的作用与微分质量公式 (114) 中的能量相同。

(ii) H is the canonical energy which generates evolution along the Killing time.

(ii) H 是沿 Killing 时间生成演化的正则能量。

(iii) Q in (116) is a Noether charge analogous to Wald’s charge.

(iii) (116) 中的 Q 是类似 Wald 荷的诺特荷。

(iv) The entropy of a black hole in the presence of non-minimal couplings is

(iv) 存在非最小耦合时黑洞的熵为

$$S = S^{BH} - Q. \tag{120}$$

To give a typical illustration of non-minimal coupling, consider the scalar theory

为了举例说明非最小耦合，我们考察标量场论

$$I[\phi, g] = -\frac{1}{2}\int d^4x\sqrt{-g}\left[(\nabla\phi)^2 + \xi R\phi^2 + m^2\phi^2\right]. \tag{121}$$

A direct check yields [28]

直接验证可得 [28]

$$Q = 2\pi\xi\int_{\mathcal{B}} d^2y\phi^2(y). \tag{122}$$

Other examples of non-minimal couplings with nontrivial charges are present for fields with integer spins.

对于整数自旋场，也存在其他带有非平凡荷的非最小耦合例子。

The quantum version of (119) should be the following formula for the average energy (117):

式 (119) 的量子形式应为平均能量 (117) 的如下公式:

$$E(\beta) = \langle HHI|\hat{H}|HHI\rangle - \beta\langle HHI|\hat{Q}|HHI\rangle. \tag{123}$$

By comparing (116) with (112), one concludes that

对比 (116) 与 (112)，可得

$$B = -\beta\langle\hat{Q}\rangle, \tag{124}$$

where $\langle\widehat{Q}\rangle = \langle HHI\left|\widehat{Q}\right|HHI\rangle$. As can be seen from (122), the expectation value $\langle\widehat{Q}\rangle$ is determined by two-point correlators, like $\left\langle\widehat{\phi}(x)\widehat{\phi}(x')\right\rangle$ in the limit $x \to x'$. Since such limits result in UV divergent terms, it is assumed that $\langle\widehat{Q}\rangle$ is a renormalized quantity.

其中 $\langle\widehat{Q}\rangle = \langle HHI\left|\widehat{Q}\right|HHI\rangle$ 。由式 (122) 可见，期待值 $\langle\widehat{Q}\rangle$ 由两点关联量决定，例如极限 $x \to x'$ 下的 $\left\langle\widehat{\phi}(x)\widehat{\phi}(x')\right\rangle$ 。由于这类极限会产生紫外发散项，因此假定 $\langle\widehat{Q}\rangle$ 是一个已重整化的量。

The presented arguments which lead to (124) cannot be considered as a rigorous proof, but they pass nontrivial checks which we consider in the next section.

得到式 (124) 的推导过程不能算作严格证明，但该结论通过了我们在下一节讨论的非平凡检验。

Generalized Black Hole Entropy and Its Interpretations

广义黑洞熵及其诠释

As follows from (112) and (124), the generalized entropy of a quantum black hole is the sum of three terms

由式 (112) 和 (122) 可得，量子黑洞的广义熵是三项之和

$$S_{\text{gen}}(\beta) = \bar{S}^{BH}(\beta) + S_q(\beta) - \langle\widehat{Q}\rangle. \tag{125}$$

The quantum correction to black hole entropy includes contribution from nonminimal couplings $\langle\widehat{Q}\rangle$. This fact has been first pointed out in [70]. The origin of $\langle\widehat{Q}\rangle$ in (125) can be traced to the delta function-like term in the scalar curvature on $\mathcal{B}$, which appears in the off-shell approach according to Eq. (64). Let us emphasize that $\bar{S}^{BH}(\beta)$ in (125) depends on the renormalized gravitational couplings, and UV divergences in $S_q(\beta)$ and $\langle\widehat{Q}\rangle$ are subtracted.

黑洞熵的量子修正包含非最小耦合的贡献 $\langle\widehat{Q}\rangle$ 。这一结论最早在文献 [70] 中提出。式 (125) 中 $\langle\widehat{Q}\rangle$ 的来源可以追溯到 $\mathcal{B}$ 标量曲率中的类 δ 函数项，根据式 (64)，该项出现在离壳方法中。需要强调的是，式 (125) 中的 $\bar{S}^{BH}(\beta)$ 依赖于重整化后的引力耦合，且 $S_q(\beta)$ 和 $\langle\widehat{Q}\rangle$ 中的紫外发散已被减除。

The consistency of (125) implies that divergences in $S_q(\beta)$ and $\langle\widehat{Q}\rangle$ are absorbed in the course of renormalization of the bare couplings in $\bar{S}^{BH}(\beta)$. To demonstrate this in the one-loop approximation, we focus on properties and interpretation of $S_q(\beta)$. To derive $S_q(\beta)$ by using (113), the one-loop effective action $W_{\text{ren}}(\beta)$ has to be considered off-shell, at β slightly different from the Hawking temperature. We denote the corresponding background manifold $\mathcal{M}_\beta$. Near the Euclidean horizon, $\mathcal{M}_\beta$ has the structure $\mathcal{C}_\beta \times \mathcal{B}$, where $\mathcal{C}_\beta$ is a two-dimensional cone with metric

式 (125) 的自洽性意味着 $S_q(\beta)$ 和 $\langle\widehat{Q}\rangle$ 中的发散会在 $\bar{S}^{BH}(\beta)$ 裸耦合的重整化过程中被吸收。为了在单圈近似下证明这一点，我们重点讨论 $S_q(\beta)$ 的性质与诠释。要利用式 (113) 推导 $S_q(\beta)$，必须在离壳条件下、当 β 与霍金温度存在微小偏差时考虑单圈有效作用量 $W_{\mathrm{ren}}(\beta)$。我们将对应的背景流形记为 $\mathcal{M}_\beta$。在欧几里得视界附近，$\mathcal{M}_\beta$ 具有结构 $\mathcal{C}_\beta\times\mathcal{B}$，其中 $\mathcal{C}_\beta$ 是一个度规为 [原文此处未结束，保留结构] 的二维锥

$$dl^2=\kappa^2\rho^2d\tau^2+d\rho^2,\ 0\le\tau<\beta. \tag{126}$$

$\mathcal{C}_\beta=R^2$, if $\beta=2\pi/\kappa$. The nice property of elliptic operators that their spectral functions are well defined on manifolds with conical singularities.

若 $\beta=2\pi/\kappa$，则 $\mathcal{C}_\beta=R^2$。椭圆算子的优良性质在于其谱函数在带锥奇点的流形上是良好定义的。

Conical singularities modify results known for regular manifolds. In particular, the heat coefficients of the heat trace (89) on $\mathcal{M}_\beta$ with p even look as follows:

锥奇点会改变规则流形上的已知结果。特别是，当 p 为偶数时，$\mathcal{M}_\beta$ 上热迹 (89) 的热系数形式如下:

$$a_p(P_E)=A_p+B_p, \tag{127}$$

where A_p are defined by standard expressions on the regular domain of $\mathcal{M}_\beta$. Additional terms $B_p, p=2k$, are some curvature polynomials on $\mathcal{B}$ which have nontrivial dependence on β. For example [34],

其中 A_p 由 $\mathcal{M}_\beta$ 规则域上的标准表达式定义，附加项 $B_p, p=2k$ 是 $\mathcal{B}$ 上的曲率多项式，对 β 存在非平凡依赖。例如文献 [34] 给出

$$B_2(\beta)=f(\beta)\mathcal{A},\ B_4(\beta)=\int_{\mathcal{B}}\sqrt{\gamma}d^2y\left(g_1(\beta)R+g_1(\beta)R_{ii}+g_3(\beta)R_{ijij}\right),$$

(128)

where $f(\beta), g_k(\beta)$ are some functions analytic at $\beta=2\pi/k$, and curvatures R_{ii}, R_{ijij} are defined by (68). The definition (113) implies that there must exist a divergent part of S_q determined by the divergent part of the effective action. In four dimensions, $n=4$,

其中 $f(\beta), g_k(\beta)$ 是 $\beta=2\pi/k$ 处解析的函数，曲率 R_{ii}, R_{ijij} 由式 (68) 定义。式 (113) 的定义表明，S_q 一定存在由有效作用量发散部分决定的发散项。在四维中，$n=4$

$$S_{q,\,\mathrm{div}}(\delta)=\left(\beta\partial_\beta-1\right)W_{\mathrm{div}}\simeq s_1\delta+s_2\ln\delta+O\left(\delta^{-1}\right), \tag{129}$$

$$s_1=-\frac{\eta}{2}\left(\beta\partial_\beta-1\right)B_2|_{\beta=\beta_H}, s_2=\eta\left(\beta\partial_\beta-1\right)B_4|_{\beta=\beta_H}. \tag{130}$$

To get (129), we used (90).

要得到式 (129)，我们用到了式 (90)。

Note that the leading divergence of S_q is connected with B_2 , and it is proportional to the horizon area $\mathcal{A}$. This divergence behaves as the Bekenstein-Hawking entropy, and, as has been suggested in [15, 75], it can be removed by the standard renormalization of the Newton constant.

请注意，S_q 的领头发散与 B_2 相关，且正比于视界面积 $\mathcal{A}$ 。该发散的行为与贝肯斯坦-霍金熵一致，正如文献 [15,75] 所提出的，它可以通过牛顿常数的标准重整化消除。

Consider a standard renormalization procedure in the one-loop effective action. To remove the divergences, the bare gravity action should be taken in form (67) with bare couplings G^B, Λ^B, c_i^B . The relation between bare and renormalized couplings G, Λ, c_i is (91). Given (91), one can show that the following formula holds [29, 40 ,] 70, 71]:

我们来考虑单圈有效作用量中的标准重整化过程。为消除发散，裸引力作用量应取形式 (67)，其裸耦合为 G^B, Λ^B, c_i^B 。裸耦合与重整化耦合 G, Λ, c_i 的关系为式 (91)。根据式 (91) 可以证明，下式成立 [70, 71]: [29, 40 ,]

$$S^{BH}(G, c_i) = S^{BH}\left(G^B, c_i^B\right) + S_{q,\,\mathrm{div}}(\delta) - \langle \widehat{Q}(\delta) \rangle_{\mathrm{div}} . \tag{131}$$

The result holds for spins 0, 1/2, and 1. Formula (131) is a nontrivial check which supports (125).

该结果适用于自旋 0、1/2 和 1 的情况。式 (131) 是一个支持式 (125) 的非平凡验证。

One of the interpretations of S_q is related to quantum entanglement in the HHI state since states on $\sum_L$ part of the Einstein-Rosen bridge are not accessible for observers in the right part of the eternal black hole, see Fig. 1. Consider a quantum system in a state $|\psi\rangle$. Suppose some part A of the system cannot be measured. A may be, for example, a spatial region. Let B be a supplement of A . Averages of operators located in B can be written as

S_q 的其中一种解释与 HHI 态中的量子纠缠有关，因为永久黑洞右半部分的观察者无法观测到爱因斯坦-罗森桥 $\sum_L$ 部分的态，参见图 1。考虑处于态 $|\psi\rangle$ 的一个量子系统。假设该系统的某一部分 A 无法被测量，例如 A 可以是一个空间区域。令 B 为 A 的补区域。位于 B 的算符的平均值可以写为

$$\langle \psi | \widehat{\mathcal{O}} | \psi \rangle = \mathrm{Tr}_B \left(\hat{\rho} \widehat{\mathcal{O}} \right), \tag{132}$$

$$\hat{\rho} = \mathrm{Tr}_A \, |\psi\rangle\langle\psi| \, . \tag{133}$$

The density matrix $\hat{\rho}$ allows different measures of the information loss about states located in A . For example, one can define the Renyi entropy

密度矩阵 $\hat{\rho}$ 可用于不同方式度量位于 A 的态的信息损失。例如，我们可以定义雷尼熵

$$S^{(\alpha)} = \frac{\mathrm{Tr}_B \hat{\rho}^{\alpha}}{1-\alpha}, \alpha > 0, \alpha \neq 1, \tag{134}$$

and the entropy of entanglement

以及纠缠熵

$$S_{\text{ent}} = \lim_{\alpha\to 1} S^{(\alpha)} = -\text{Tr}_B \hat{\rho} \ln \hat{\rho}. \tag{135}$$

A review of quantum entanglement and entanglement measures can be found in [51]. Entanglement in quantum field theories is discussed, e.g., in [17].

量子纠缠和纠缠度量的综述可见文献 [51]。量子场论中的纠缠讨论可见例如文献 [17]。

In the considered case of an eternal black hole, A and B are two sides of the Einstein-Rosen bridge separated by the bifurcation surface $\mathcal{B}$, say, $B = \sum_R$ and $A = \sum_L$. To calculate the entanglement entropy, we use representation (83) for the HHI state. If one assumes that $\langle HHI \mid HHI \rangle = 1$, the normalization factor in (83) is $N = \text{Tr}\, e^{-\beta_H :\hat{H}:} = Z(\beta_H)$. This yields for natural parameters n

在我们讨论的永久黑洞情形中，A 和 B 是爱因斯坦-罗森桥被分岔面 $\mathcal{B}$ 分隔开的两个侧边，分别为 $B = \sum_R$ 和 $A = \sum_L$ 。为计算纠缠熵，我们对 HHI 态使用表示 (83)。若假设 $\langle HHI \mid HHI \rangle = 1$ ，则式 (83) 中的归一化因子为 $N = \text{Tr}\, e^{-\beta_H :\hat{H}:} = Z(\beta_H)$ 。由此可得自然参数 n 满足

$$\text{Tr}_B \hat{\rho}^n = \frac{Z(n\beta_H)}{Z^n(\beta_H)} = \exp\left(-W(n\beta_H) + nW(\beta_H)\right), \tag{136}$$

where $W(\beta)$ should be considered as a regularized action. To get (136) we used (100) and noticed mutual cancelation of the vacuum energies in $W(n\beta_H)$ and in $nW(\beta_H)$. Since $W_{\text{ren}}(\beta)$ can be defined at arbitrary β , one can replace n in the r.h.s. of (136) with a parameter $\alpha > 0$ and use (134) and (135) to see that

其中 $W(\beta)$ 应视为正则化作用量。为得到式 (136)，我们使用了式 (100)，并注意到真空能在 $W(n\beta_H)$ 和 $nW(\beta_H)$ 中相互抵消。由于 $W_{\text{ren}}(\beta)$ 可以在任意 β 处定义，我们可以将式 (136) 右侧的 n 替换为参数 $\alpha > 0$ ，再利用式 (134) 和 (135) 即可得到

$$S_{\text{ent}} = S_q. \tag{137}$$

That is, the entanglement entropy in the HHI state is given by formula (113) and is a part of the generalized entropy (125).

也就是说，HHI 态中的纠缠熵由式 (113) 给出，它是广义熵 (125) 的一部分。

The fact that S_{ent} is proportional to the horizon area and can be related to the Bekenstein-Hawking entropy was first pointed out in [12, 31, 73] . Since computations of the entanglement entropy based on (113) are reduced to finding effective actions on manifolds with conical singularities, S_{ent} was also called a geometric entropy. Pioneering studies of S_{ent} can be found in [54-57]. A separate interesting topic is the entanglement entropy of gauge fields which is discussed in a number of publications, see, e.g., [19,23]. A comprehensive review of entanglement entropy of black holes is [71].

S_{ent} 正比于视界面积且可与贝肯斯坦-霍金熵联系起来，这一结论最早由 [12, 31, 73] 指出。由于基于式 (113) 的纠缠熵计算可以归约为寻找带锥奇异流形上的有效作用量，S_{ent} 也被称为几何熵。S_{ent} 的开创性研究可见文献 [54-57]。规范场的纠缠熵是一个独立的有趣课题，已有诸多文献讨论，例如可见 [19,23]。黑洞纠缠熵的综合性综述可见文献 [71]。

One can also come to (137) with the help of (83) by noticing that the reduced density matrix for the HHI state is thermal, $\hat{\rho} = e^{-\beta_H :\hat{H}:}/Z(\beta_H)$. This means that the entanglement entropy coincides with the entropy of the thermal atmosphere of a black hole. The first attempts to relate the Bekenstein-Hawking entropy to the thermal atmosphere have been made in [50,84]. The thermal entropy can be derived from high-temperature asymptotic (102), and it diverges due to the infinite blue-shift of the temperature at the horizon, see (101). To avoid divergences, a narrow region near the horizon should be excluded when integrating in (102). If the cutoff is made at a proper distance equal the Planck length, the thermal entropy is of the order of the Bekenstein-Hawking entropy [50]. The divergences in the thermal entropy can be eliminated by using other regularizations, for instance, the Pauli-Villars (PV) regularization [22]. Equivalence between divergences of thermal entropy and entanglement entropy in different regularizations is discussed in [29].

借助式 (83) 也能得到式 (137)，只要注意到 HHI 态的约化密度矩阵是热态，$\hat{\rho} = e^{-\beta_H :\hat{H}:}/Z(\beta_H)$。这意味着纠缠熵与黑洞热大气的熵相等。最早尝试将贝肯斯坦-霍金熵与热大气联系起来的工作是文献 [50,84]。热熵可由高温渐近式 (102) 导出，由于视界处温度存在无穷蓝移，热熵会发散，见式 (101)。为了避免发散，对 (102) 积分时应当排除视界附近的一个狭窄区域。若在普朗克长度量级的固有距离处设置截断，得到的热熵就与贝肯斯坦-霍金熵同阶 [50]。热熵的发散可以通过其他正则化方法消除，例如泡利-维拉尔 (PV) 正则化 [22]。文献 [29] 讨论了不同正则化方案中热熵发散与纠缠熵发散的等价性。

To simplify the presentation in the last sections, we restricted the discussion by static black holes. The analysis can be extended to rotating black holes, see [30] and the references therein.

为简化前几节的表述，我们仅讨论了静态黑洞。相关分析可以推广到旋转黑洞，参见文献 [30] 及其中的参考文献。

Concluding Remarks

结语

The aim of this chapter was an introduction to generalized thermodynamics of quantum black holes and related notions which appeared in the last decades. The key conclusion is that the Bekenstein-Hawking entropy is a part of the generalized entropy (125) of a black hole, and it is deeply related to quantum effects via the renormalization of the ultraviolet divergences.

本章旨在介绍近几十年来出现的量子黑洞广义热力学及相关概念。核心结论是，贝肯斯坦-霍金熵是黑洞广义熵 (125) 的一部分，它通过紫外发散的重整化与量子效应密切相关。

Renormalization requires the bare entropy which does not have any statistical meaning in the perturbative quantum gravity. The problem of the bare entropy in (131) can be resolved [53] if the Einstein gravity is entirely induced by quantum effects, so that (125) becomes $S_{\text{gen}} = S_q - \langle \hat{Q} \rangle$. This mechanism can be checked at a one-loop order in QFT models where the leading ultraviolet divergences are canceled out [28, 33] . The idea of induced gravity was formulated long ago [66,67], see [78] for a review, but it does not contradict with a modern perspective. From the point of view of the open string theory, black hole entropy can be considered as a loop effect [48], in full analogy with its origin in induced gravity. There still remains the problem of statistical interpretation of the ”contact” term $\langle \hat{Q} \rangle$ in the entropy, whose presence is unavoidable to compensate for divergences in S_q .

重整化需要裸熵，而裸熵在微扰量子引力中不具备任何统计意义。如果爱因斯坦引力完全由量子效应诱导产生，那么 (131) 中的裸熵问题就可以得到解决 [53]，此时 (125) 变为 $S_{\text{gen}} = S_q - \langle \hat{Q} \rangle$ 。该机制可以在量子场论模型的单圈阶得到验证， 在这些模型中领头阶紫外发散被抵消 [28, 33] 。诱导引力的思想早在很久以前就被提出 [66,67]，综述可参考 [78]，它与现代观点并不矛盾。从开弦理论的角度看，黑洞熵可以被视为圈效应 [48]， 这和它在诱导引力中的起源完全一致。熵中” 接触项” $\langle \hat{Q} \rangle$ 的统计解释问题至今仍未解决， 该项对于抵消 S_q 中的发散是必不可少的。

We have not discussed alternative interpretations of the black hole entropy which go beyond the perturbative quantum gravity. A number of promising ideas to statistical explanation of the Bekenstein-Hawking formula are reviewed in [18]. Among them is a remarkable calculation of the entropy of extremal [74] and near-extremal black holes in string theory, more on that can be found in [63].

我们尚未讨论超出微扰量子引力范畴的黑洞熵替代解释。[18] 综述了许多解释贝肯斯坦-霍金公式统计起源的可行思路，其中包括弦理论中对极端 [74] 和近极端黑洞熵的经典计算，更多相关内容可查阅 [63]。

There is a mounting evidence that the Bekenstein-Hawking entropy formula can be applied for systems which are not black holes. For instance, in higher dimensional anti-de Sitter gravities, it allows successful holographic description [60, 65] of the entanglement entropy in dual conformal field theories. In a similar way, $\mathcal{A}/(4G)$ can be interpreted as an entanglement entropy in low-energy limit of quantum gravity for regions restricted by minimal surfaces of the area $\mathcal{A}$ [37, 58] .

越来越多的证据表明，贝肯斯坦-霍金熵公式可以应用于非黑洞系统。例如， 在高维反德西特引力中，该公式可以成功对对偶共形场论的纠缠熵给出全息描述 [60, 65]。类似地，$\mathcal{A}/(4G)$ 可以解释为量子引力低能极限下，面积为 $\mathcal{A}$ [37, 58] 的极小曲面所围区域的纠缠熵。

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